Time-change Method in Quantitative Finance

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Zhenyu Cui

Abstract

In this thesis I discuss the method of time-change and its applications in quantitative finance.

I mainly consider the time change by writing a continuous diffusion process as a Brownian motion subordinated by a subordinator process. I divide the time change method into two cases: deterministic time change and stochastic time change. The difference lies in whether the subordinator process is a deterministic function of time or a stochastic process of time.

Time-changed Brownian motion with deterministic time change provides a new viewpoint to deal with option pricing under stochastic interest rates and I utilize this idea in pricing various exotic options under stochastic interest rates.

Time-changed Brownian motion with stochastic time change is more complicated and I give the equivalence in law relation governing the "original time" and the "new stochastic time" under different clocks. This is readily applicable in pricing a new product called "timer option". It can also be used in pricing barrier options under the Heston stochastic volatility model.

Conclusion and further research directions in exploring the ideas of time change method in other areas of quantitative finance are in the last chapter.

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Dedicated to my Parents: Jianjun Cui and Ping Fan who raised me
Dedicated also to the awesome God Jesus Christ, who always love me

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Chapter 1

Introduction

In this chapter, I will briefly review the literature on applying time-change method to the area of quantitative finance. By the "time-change" method, we mean to view or build financial models by making use of time-changed stochastic processes. We let $X_t, t \geqslant 0$ denote a stochastic process and let $T_s, s \geqslant 0$ denote a non-negative and non-decreasing stochastic process. Usually we also require that $lim_{s\to\infty} T_s = \infty$. Then the *time-changed process* is defined as $Y_s, s \geqslant 0$ where $Y_s = X_{T_s}$. Note that here we do not assume that X_s and T_s are independent. We say that X_t evolves under the *operational time* and T_s is the time change, stochastic clock or *business time*.

"Time change", especially "time-changed Brownian motion" is a standard tool in probability theory and has been used in Ito and Mckean(1965) to transform the study of diffusions to the study of Brownian motions.

Dambis(1965), Dubins and Schwartz (1965) independently showed that

Any continuous martingale is a time-changed Brownian motion

Monroe (1978) extended the above result to a very general setting

Any semi-martingale is a time-changed Brownian motion

The generality of the Monroe(1978) Theorem allows us to apply the time-change technique to many problems in finance, because most stochastic processes used in finance are semi-martingales.

1.1 Time change and economic foundation

The idea of using time change idea in finance can be traced back to Clark (1973). The problem he was interested in was to model cotton futures price data and he used the time change to explain the non-normality of observed returns. He gave the price as $S(t) = W(X(t)), t \geqslant 0$ and the economic interpretation of $X(t)$ is the cumulative volume of traded contracts. Later Ane and Geman (2000) did an empirical study and showed that in a general non-parametric setting, in order to recover the normality of stock returns, the stock price should be represented as $S(t) = W(T(t)), t \geqslant 0$ and here $T(t)$ is the transaction clock and represents the number of trades. This is a powerful representation because it represents the complex phenomenon into a single entity. Thus $T(t)$ contains all the information of the empirical property of the asset price in this economy, i.e, the skewness and fat tail phenomenon can be recovered by specifying different $T(t)$ processes.

1.2 Time change techniques in the literature

"Time change technique", at its origin, is a popular and powerful technique in probability theory. Ito and Mckean (1965) showed that we can replace the study of diffusions to the study of Brownian motions through a proper time change. Later Williams (1974) showed that we could represent geometric Brownian motion as a time-changed Bessel process, which is usually called the "Lamperti Relation". Based on this result, Geman and Yor (1993) gave the Laplace transform of the price of the Arithmetic Asian option in the Black-Scholes framework.

This thesis does not focus on the empirical performance of time-change representation of stock prices or its economic interpretations. It rather focuses on the application of the time-change techniques in derivatives pricing under various frameworks other than the Black-Scholes one.

1.3 Contribution and outline of the Thesis

The major contribution of the thesis is to establish the link between the time under the original clock and the time under the new stochastic clock when the subordinator is an integrated diffusion process. This is given in the following theorem

Theorem 1.3.1. *Assuming the general dynamic for the diffusion process* V_t

$$dV_t = \mu(V_t)\,dt \,+\, v(V_t)\,dW_t \tag{1.1}$$

$\mu(.)$ *and* $v(.)$ *are continuous and smooth functions such that they have derivatives up to any order. Let* $\tau = \inf\{t; \int_0^t \lambda^2(V_s)ds = \eta\} \in (0,\infty)$ *be the first passage time of the integrated functional of* V_s *to a fixed level* $\eta \in (0,\infty)$*, here* $\lambda(.)$ *is a deterministic function such that* $\int_0^\infty \lambda^2(V_s)ds \to +\infty$*, then the law of* (τ, V_τ) *is given by*

$$(\tau, V_\tau) \;\sim^{law}\; \left(\int_0^\eta \frac{1}{\lambda^2(X_s)}ds,\; X_\eta\right) \tag{1.2}$$

where X_t *is uniquely governed by the SDE*

$$\begin{cases} df(X_t) &=& \frac{h(X_t)}{\lambda^2(X_t)}dt + dB_t, \\ X_0 &=& V_0 \end{cases} \tag{1.3}$$

where B *is a standard Brownian motion,* f *and* h *are functions given in equations* (1.5) *and* (1.6) *in the proof.*

Proof

Suppose $\theta(t) = \int_0^t \lambda^2(V_s)ds < \infty$ for any $t < \infty$. $\theta(t)$ is a continuous function on $\mathbb{R}$.

Let $\tau(y)$ be the inverse function of $\theta(t)$ for $t \in (0,\infty)$, i.e. $\tau(\theta(t)) = t$.

Since $\theta(\tau(y)) = y$, thus by the Inverse Function Theorem, we have $\theta'(\tau(y)) = \frac{1}{\tau'(y)}$ and $\theta'(s) = \lambda^2(V_s)$, therefore

$$\begin{aligned} \tau(y) &= \int_0^y \tau'(s)ds = \int_0^y \frac{1}{\theta'(\tau(s))}ds = \int_0^y \frac{1}{\lambda^2(V_{\tau(s)})}ds \text{ and} \\ \theta'(\tau(y)) &= \frac{1}{\tau'(y)} = \lambda^2(V_{\tau(y)}) \end{aligned} \tag{1.4}$$

We choose the function

$$f(s) = \int_0^s \frac{\lambda(z)}{v(z)} dz \tag{1.5}$$

and then define

$$h(s) = \mu(s) f^{'}(s) + \frac{1}{2} v^2(s) f^{''}(s) \tag{1.6}$$

By Ito's lemma, we have

$$\begin{aligned} df(V_t) &= h(V_t)dt + f^{'}(V_t)v(V_t)dW_t \\ &= h(V_t)dt + \lambda(V_t)dW_t \end{aligned} \tag{1.7}$$

Define $\tau(t) = \inf\{u > 0, \int_0^u \lambda^2(V_s)ds = t\}$, which is the inverse of $\theta(t)$. It is well-defined, because for any fixed t, we have $P(\int_0^\infty \lambda^2(V_s)ds > t) = 1$. Then by the Dubins-Schwartz theorem(Karatzas and Shreve(1991), p174), we have

$$M(\tau(t)) = \int_0^{\tau(t)} \lambda(V_s)dW_s = B_t \tag{1.8}$$

Now if we integrate both sides of the equation (1.7) from 0 to $\tau(t)$, we have

$$\int_0^{\tau(t)} \lambda(V_s)dW_s = f(V_{\tau(t)}) - f(V_0) - \int_0^{\tau(t)} h(V_s)ds \tag{1.9}$$

From equation (1.8) and (1.9), we have

$$B_t = \int_0^{\tau(t)} \lambda(V_s)dW_s = f(V_{\tau(t)}) - f(V_0) - \int_0^{\tau(t)} h(V_s)ds \tag{1.10}$$

Then we take the differentiate representation of (1.10) and have

$$df(V_{\tau(t)}) = h(V_{\tau(t)})\tau_t^{'} dt + dB_t \tag{1.11}$$

Also from equation (1.4), we have $\tau(t) = \int_0^t \frac{1}{\lambda^2(V_{\tau_s})} ds$, so $\tau(t)^{'} = \frac{1}{\lambda^2(V_{\tau(t)})}$. Then we can rewrite (1.11) as

$$df(V_{\tau(t)}) = \frac{h(V_{\tau(t)})}{\lambda^2(V_{\tau(t)})} dt + dB_t \tag{1.12}$$

Then if we let $X_t = V_{\tau(t)}$, we finally have

$$(\tau, V_\tau) \longleftrightarrow^{law} \left(\int_0^\eta \frac{1}{\lambda^2(X_s)} ds, X_\eta \right) \tag{1.13}$$

where X_t is governed by

$$df(X_t) = \frac{h(X_t)}{\lambda^2(X_t)} dt + dB_t, X_0 = V_0 \tag{1.14}$$

□

When the time under the stochastic clock is η, we can see that (1.13) gives the equivalence in law relation of both the original time τ and also the process value at that time τ. Thus we can simulate τ measured by the original clock and also V_τ if we know the *current time* under the new stochastic clock. This constructs a one to one correspondence of the "new time" with the "old time". The study of this relationship between the two times under the stochastic time change is motivated by the study of an exotic derivative called "timer option". In chapter 5, we will discuss the pricing of this product and Theorem 1.3.1 here will play the vital role in the study.

In the above we have discussed stochastic time change where the time change subordinator is an integrated diffusion process. A similar problem is how to link the "new time" to the "old time" for deterministic time change. This problem is actually easier because you can directly invert the function relating the two times. For example, if the deterministic time change is $\xi(t) = \int_0^t f(s)ds$ where $f(.)$ is a deterministic function $f : \mathbb{R} \to \mathbb{R}$. Then the two times are related as $t_2 = \xi^{-1}(t_1)$ where t_1 is the new time and t_2 is the old time.

Although simple, the deterministic time change can be used in problems related to stochastic interest rates. The idea is that when you time-change a diffusion process to be a time-changed Brownian motion with deterministic time change, you can use the standard results of Brownian motion available, such as the distribution of the first passage time of the Brownian motion, the distribution of the excursion time of the Brownian motion above a fixed level. The idea of using a deterministic time change can provide us new insights into some classical problems, such as pricing options under stochastic interest rates. In Chapter 3, we will apply this method to the pricing of the standard European option, the chooser option, the forward-start option, and the compound option under stochastic interest rates.

Using deterministic time change ideas to price path-dependent options under stochastic interest rates is successfully implemented in Bernard et al (2008) for pricing the continuous barrier options under the Vasicek stochastic interest rate model. In Chapter 4, we push this idea even further and consider the pricing of standard Parisian option under Vasicek stochastic interest rate model.

To summarize, in this thesis I have used the deterministic time change technique in pricing exotic options under stochastic interest rates. I also explore the linkage between the old time and the new time under the stochastic time change and use it to price a new exotic product "timer option". I also make use of the stochastic time-change method to price the continuous barrier options under the Heston stochastic volatility model.

The organization of the thesis is as follows: each chapter is an independent research article that I write and they are categorized under the general theme of "using time-change method in quantitative finance". Chapters 2, 3 and 4 utilize the deterministic time change idea. Chapters 5 and 6 utilize the stochastic time change idea. Chapter 7 is the conclusion of the thesis and discusses further research directions.

Chapter 2

Option pricing under Merton's short rate model

2.1 Introduction

In this chapter we use the "change of numeraire" technique and the deterministic time change idea to provide a simpler derivation of the price of a European call option under Merton's model of the short rate. We show that the closed-form formula in Kung and Lee (2009) is wrong and arrive at different conclusions from Kung and Lee (2009) based on our numerical results.

2.2 Derivation

Consider the following dynamics as in Merton (1973) under the risk-neutral measure Q (Our notation S_t is equivalent to $P_s(t)$ in Kung and Lee (2009))

$$dr(t) = \alpha dt + \sigma dZ_1(t)$$
$$\frac{dS_t}{S_t} = r(t)dt + \sigma_s \left[\rho dZ_1(t) + \sqrt{1-\rho^2} dZ_2(t)\right] \tag{2.1}$$

where $Z_1(t)$ and $Z_2(t)$ are two independent Brownian motions.

Let $P_b(t,T)$ be the price at time t of a risk-free zero-coupon bond which pays 1 dollar at time T. Then (see Merton (1973)), with $\tau = T - t$,

$$P_b(t,T) = \exp\{-r(t)\tau - \frac{\alpha}{2}\tau^2 + \frac{\sigma^2\tau^3}{6}\} \tag{2.2}$$

The instantaneous forward rate $f(t,T) = -\frac{\partial \ln P_b(t,T)}{\partial T}$ satisfies,

$$df(t,T) = \sigma^2(T-t)dt + \sigma dZ_1(t) \tag{2.3}$$

By definition, $P_b(t,T) = \exp(-\int_t^T f(t,u)du)$, thus by Ito's Lemma,

$$\frac{dP_b(t,T)}{P_b(t,T)} = r(t)dt - \sigma(T-t)dZ_1(t) \tag{2.4}$$

Given (2.1) and (2.4), denote $\sigma_P(t,T) = \sigma(T-t)$. By Ito's lemma,

$$S_T = S_t \exp\left(\int_t^T r(u)du - \frac{1}{2}\sigma_s^2(T-t) + \int_t^T \rho\sigma_s dZ_1(u) + \int_t^T \sqrt{1-\rho^2}\sigma_s dZ_2(u)\right) \tag{2.5}$$

$$P_b(T,T) = P_b(t,T)\exp\left(\int_t^T r(u)du - \frac{1}{2}\int_t^T \sigma_P^2(u,T)du - \int_t^T \sigma_P(u,T)dZ_1(u)\right) \tag{2.6}$$

Now similar to Bernard et al (2008), we treat $P_b(t,T)$ as our numeraire and use the martingale property of the relative price $S_T/P(T,T)$ to write the dynamics under the T-forward-neutral measure Q^T

$$S_T = \frac{S_t}{P_b(t,T)}\exp\left(\int_t^T (\sigma_P(u,T)+\rho\sigma_s)dZ_1^T(u) + \int_t^T (\sigma_s\sqrt{1-\rho^2})dZ_2^T(u)\right.$$
$$\left. -\frac{1}{2}\int_t^T \left((\sigma_P(u,T)+\rho\sigma_s)^2 + \sigma_s^2(1-\rho^2)\right)du\right) \tag{2.7}$$

where $dZ_1^T(t) = dZ_1(t) + \sigma_P(t,T)dt$ and $dZ_2^T(t) = dZ_2(t)$.

We can calculate

$$\xi(t,T) = \int_t^T \left((\sigma_P(u,T)+\rho\sigma_s)^2 + \sigma_s^2(1-\rho^2)\right)du = \frac{\sigma^2}{3}\tau^3 + \rho\sigma\sigma_s\tau^2 + \sigma_s^2\tau \tag{2.8}$$

Then, at time T, under Q^T,

$$S_T = \frac{S_t}{P_b(t,T)}\exp(B_{\xi(t,T)} - \frac{1}{2}\xi(t,T)) \tag{2.9}$$

Let $C_{BS}(S_0,K,r,T,\sigma)$ denote the price of a call option in the Black-Scholes setting where the initial underlying price is S_0, strike is K, interest rate is r, maturity is T and volatility is σ. Then from equation (2.9),

$$C(t) = P_b(t,T)\{E^{Q^T}\left[(S_T-K)^+\right]\} = P_b(t,T)C_{BS}\left(\frac{S_t}{P_b(t,T)}, K, 0, T-t, \sqrt{\frac{\xi(t,T)}{T-t}}\right)$$
$$= P_b(t,T)\left(\frac{S_t}{P_b(t,T)}\mathcal{N}(\frac{\ln(\frac{S_t}{KP_b(t,T)}) + \frac{1}{2}\xi(t,T)}{\sqrt{\xi(t,T)}}) - K\mathcal{N}(\frac{\ln(\frac{S_t}{KP_b(t,T)}) - \frac{1}{2}\xi(t,T)}{\sqrt{\xi(t,T)}})\right)$$
$$= S_t\mathcal{N}\left(\frac{\ln(\frac{S_t}{KP_b(t,T)}) + \frac{1}{2}\xi(t,T)}{\sqrt{\xi(t,T)}}\right) - KP_b(t,T)\mathcal{N}\left(\frac{\ln(\frac{S_t}{KP_b(t,T)}) - \frac{1}{2}\xi(t,T)}{\sqrt{\xi(t,T)}}\right) \tag{2.10}$$

where $P_b(t,T)$ and $\xi(t,T)$ are given in equation (2.2) and equation (2.8).

Notice that in Kung and Lee (2009), the notation Σ they use is equivalent to our notation $\xi(t,T)$, but the equation (30) in their paper is quite different from the equation (2.10) here.

We use the call price obtained from crude Monte Carlo as the benchmark (M=1, 000, 000 runs) and have the following numerical comparison (see Table 2.1) based on the parameter values used in Kung and Lee (2009).

In Table 2.1, $call_{MC}$ represents the price obtained using crude Monte Carlo and is our benchmark. $call_{our-formula}$ is the price obtained using our formula (2.10). $call_{K-L}$ is the price obtained using the formula in Kung and Lee (2009).

2.3 Conclusion of Chapter 2

We see that our formula gives results that are closer to the true Monte Carlo prices and any discrepancy is due to Monte Carlo error. For certain parameters, the formula in Kung and Lee (2009) gives results that are far from the true price. Thus the formula (30) of Kung and Lee (2009) is incorrect and the correct one is given in (2.10) here. From the numerical results, we conclude that in a stochastic interest rates model, the Black-Scholes formula appears to **undervalue** out-of-the-money, at-the-money and in-the-money calls. This is intuitive since the premium of the call option should be higher because of the added randomness introduced by adding stochastic interest rates.

Table 2.1: Prices for calls when the short rate at initial time t is 0.06.

Panel A Out-of-the-money calls, $P_s = 15$, $X = 20$, $\sigma_s = 0.3$

τ	ρ	α	σ	$call_{MC}$	$call_{our-formula}$	$call_{K-L}$	$call_{BS}$
0.25	0.2	0.002	0.02	0.0357	0.0355	0.0345	0.0351
	0.2	0.002	0.08	0.0368	0.0366	0.0330	0.0351
	0.2	0.008	0.02	0.0356	0.0356	0.0346	0.0351
	0.2	0.008	0.08	0.0363	0.0367	0.0331	0.0351
	0.8	0.002	0.02	0.0360	0.0365	0.0326	0.0351
	0.8	0.002	0.08	0.0411	0.0407	0.0257	0.0351
	0.8	0.008	0.02	0.0369	0.0366	0.0327	0.0351
	0.8	0.008	0.08	0.0408	0.0408	0.0258	0.0351
0.75	0.2	0.002	0.02	0.4203	0.4204	0.4044	0.4128
	0.2	0.002	0.08	0.4440	0.4431	0.3906	0.4128
	0.2	0.008	0.02	0.4231	0.4251	0.4090	0.4128
	0.2	0.008	0.08	0.4484	0.4479	0.3951	0.4128
	0.8	0.002	0.02	0.4354	0.4372	0.3705	0.4128
	0.8	0.002	0.08	0.5118	0.5099	0.2550	0.4128
	0.8	0.008	0.02	0.4410	0.4420	0.3749	0.4128
	0.8	0.008	0.08	0.5161	0.5150	0.2587	0.4128

Panel B At-the-money calls, $P_s = 20$, $X = 20$, $\sigma_s = 0.3$

τ	ρ	α	σ	$call_{MC}$	$call_{our-formula}$	$call_{K-L}$	$call_{BS}$
0.25	0.2	0.002	0.02	1.3409	1.3442	1.3385	1.3416
	0.2	0.002	0.08	1.3487	1.3508	1.3289	1.3416
	0.2	0.008	0.02	1.3455	1.3461	1.3403	1.3416
	0.2	0.008	0.08	1.3541	1.3526	1.3308	1.3416
	0.8	0.002	0.02	1.3526	1.3501	1.3266	1.3416
	0.8	0.002	0.08	1.3737	1.3739	1.2802	1.3416
	0.8	0.008	0.02	1.3538	1.3520	1.3285	1.3416
	0.8	0.008	0.08	1.3719	1.3758	1.2821	1.3416
0.75	0.2	0.002	0.02	2.5037	2.5077	2.4797	2.4918
	0.2	0.002	0.08	2.5474	2.5450	2.4529	2.4918
	0.2	0.008	0.02	2.5284	2.5245	2.4964	2.4918
	0.2	0.008	0.08	2.5604	2.5616	2.4697	2.4918
	0.8	0.002	0.02	2.5401	2.5371	2.4189	2.4918
	0.8	0.002	0.08	2.6573	2.6577	2.1935	2.4918
	0.8	0.008	0.02	2.5445	2.5537	2.4358	2.4918
	0.8	0.008	0.08	2.6733	2.6741	2.2108	2.4918

Panel C In-the-money calls, $P_s = 25$, $X = 20$, $\sigma_s = 0.3$

τ	ρ	α	σ	$call_{MC}$	$call_{our-formula}$	$call_{K-L}$	$call_{BS}$
0.25	0.2	0.002	0.02	5.3816	5.3790	5.3772	5.3773
	0.2	0.002	0.08	5.3775	5.3809	5.3740	5.3773
	0.2	0.008	0.02	5.3777	5.3825	5.3806	5.3773
	0.2	0.008	0.08	5.3833	5.3843	5.3774	5.3773
	0.8	0.002	0.02	5.3790	5.3809	5.3735	5.3773
	0.8	0.002	0.08	5.3837	5.3886	5.3597	5.3773
	0.8	0.008	0.02	5.3829	5.3844	5.3770	5.3773
	0.8	0.008	0.08	5.3900	5.3920	5.3632	5.3773
0.75	0.2	0.002	0.02	6.3431	6.3382	6.3195	6.3227
	0.2	0.002	0.08	6.3596	6.3596	6.2981	6.3227
	0.2	0.008	0.02	6.3694	6.3645	6.3460	6.3227
	0.2	0.008	0.08	6.3870	6.3857	6.3246	6.3227
	0.8	0.002	0.02	6.3550	6.3579	6.2801	6.3227
	0.8	0.002	0.08	6.4412	6.4387	6.1423	6.3227
	0.8	0.008	0.02	6.3882	6.3841	6.3069	6.3227
	0.8	0.008	0.08	6.4669	6.4642	6.1702	6.3227

Chapter 3

Exotic Option pricing under Stochastic interest rates

In this chapter, we will propose a unified pricing method for exotic options under stochastic interest rates. This method is motivated by the paper of Bernard et al (2008). Key to our pricing method is the idea that after applying the deterministic time-change, we can represent the asset price as the exponential of time-changed Brownian motion. Then we can price options using the standard Black-Scholes theory under this "new time".

3.1 General methodology of time change

Our method is based on the time-changed Brownian motion representation of continuous martingales and the main tool is the Dubins-Schwartz Theorem.

3.1.1 Dubins Schwarz Theorem

This is the main theorem that our method relies on and we cite it here (refer to Karatzas and Shreve (1991), p174)

Theorem 3.1.1. *Let $M = \{M_t, F_t; 0 \leqslant t < \infty\} \in \mathcal{M}^{c,loc}$ be a continuous local martingale and it satisfies $\lim_{t\to\infty}\langle M\rangle_t = \infty$, a.s. Define, for each $0 \leqslant s < \infty$, the stopping time $T(s) = \inf\{t \geqslant 0; \langle M\rangle_t > s\}$. Then the time-changed process $B_s = M_{T(s)}; 0 \leqslant s < \infty$ is a standard one-dimensional Brownian motion. In particular, the filtration $\{\mathcal{G}_s\} = \{\mathcal{F}_{T(s)}\}$ satisfies the usual conditions and we have, a.s.*

$$M_t = B_{\langle M\rangle_t}; 0 \leqslant t < \infty \tag{3.1}$$

Note that we have not changed the underlying measure but have changed the underlying filtration. Basically we can express the continuous local martingale as a Brownian motion at the ***transformed time****.*

3.2 Stochastic interest rates with constant asset volatility

3.2.1 Asset and interest rates dynamics

We assume that the asset follows the lognormal dynamics correlated to the interest rates. The interest rate model considered here is driven by a unique factor, correlated to the one of the asset.

Under the risk-neutral measure Q, the dynamic of the asset A_t and of the zero-coupon bond process $P(t,T)$ are given as

$$\frac{dA_t}{A_t} \quad = r_t dt + \sigma dZ^{\mathrm{Q}}(t) \tag{3.2}$$

and

$$\frac{dP(t,T)}{P(t,T)} \quad = r_t dt - \sigma_P(t,T) dZ_1^{\mathrm{Q}}(t)$$

where $Z^{\mathrm{Q}}(t)$ and $Z_1^{\mathrm{Q}}(t)$ are standard Q-Brownian motions with correlation coefficient equal to ρ, and $\sigma_P(t,T)$ is a deterministic function specified by the particular model we are dealing with (e.g. the Hull-White model). r_t is the instantaneous short rate process.

Let us now construct a Brownian motion Z_2^{Q} independent of Z_1^{Q}. It is possible to split up Z^{Q} into the two following components $dZ^{\mathrm{Q}}(t) = \rho dZ_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathrm{Q}}(t)$. We have therefore de-correlated the pure interest rate risk from the other sources of risk. The dynamics of the asset given in (3.2) can now be reexpressed as

$$\frac{dA_t}{A_t} = r_t dt + \sigma \left(\rho dZ_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathrm{Q}}(t) \right) \tag{3.3}$$

Recall that the Radon-Nikodym density which allows us to build the forward-neutral measure Q_T is defined by

$$\frac{d\mathrm{Q}_T}{dQ} = e^{-\int_0^T \sigma_P(s,T) dZ_1^{\mathrm{Q}}(s) - \frac{1}{2}\int_0^T \sigma_P^2(s,T) ds}$$

We also define $Z_2^{\mathrm{Q}_T}$ such that $Z_1^{\mathrm{Q}_T}$ and $Z_2^{\mathrm{Q}_T}$ are non-correlated Q_T-Brownian motions. The dynamics of A_t and $P(t,T)$ under Q_T finally write as

$$\frac{dA_t}{A_t} \quad = (r_t - \sigma\rho\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{Q_T} + \sqrt{1-\rho^2}dZ_2^{Q_T}\right)$$

and

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{Q_T}$$

Upon integration from 0 to t on both sides of these two dynamics, we obtain

$$\begin{aligned} A_t = \frac{A_0}{P(0,t)} \ \exp\Bigg(&\int_0^t (\sigma_P(u,t) + \rho\sigma \ )dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2} \ dZ_2^{Q_T}(u) \\ &+ \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t) + \rho\sigma) + \frac{\sigma_P^2(u,t) - \sigma^2}{2}\right) du\Bigg) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\Bigg(&- \int_0^t (\sigma_P(u,T) - \sigma_P(u,t))\, dZ_1^{Q_T} \\ &+\frac{1}{2}\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))^2\, du\Bigg) \end{aligned} \tag{3.5}$$

Finally, note that the following dynamic will also be useful in the coming developments

$$\begin{aligned} \frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} \exp\Bigg(&\int_0^t (\sigma_P(u,T) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}dZ_2^{Q_T}(u) \\ &-\frac{1}{2}\int_0^t \left((\sigma_P(u,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)\right) du\Bigg) \end{aligned} \tag{3.6}$$

3.2.2 Asset dynamics

Now we denote

$$N_t = \int_0^t (\sigma_P(u,T) + \rho\sigma)\, dZ_1^{Q_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}dZ_2^{Q_T}(u) \tag{3.7}$$

$$\xi(t) = \int_0^t \left((\sigma_P(u,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)\right) du \tag{3.8}$$

From expression (3.8) we can see that the term inside the integral is always positive, especially that it comprises of a positive term plus a positive constant, so

$$\lim_{t\to\infty} \xi(t) = +\infty \tag{3.9}$$

Also from expression (3.7) and (3.8), we note that

$$\langle N \rangle_t = \xi(t) \tag{3.10}$$

which means the quadratic variation of N_t is just $\xi(t)$.

The facts (3.10) and (3.9) satisfy the assumptions of **Dubins-Schwarz Theorem of Time Change of Martingale**(see Karatzas and Shreve (1991), p174, Thm 3. 4. 6), which is Theorem 3.1.1 in the first section, especially note the equation (3.1). Thus by the theorem, we know that there exists a Q^T-Brownian Motion B such that

$$N_t = B_{\xi(t)}, t \in [0, T] \tag{3.11}$$

From result (3.6) and definitions (3.7), (3.8), we have

$$\frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} \exp(B_{\xi(t)} - \frac{1}{2}\xi(t)) \tag{3.12}$$

Set t=T in (3.12), we have that, under the measure Q^T

$$\frac{A_T}{P(T,T)} = A_T = \frac{A_0}{P(0,T)} \exp(B_{\xi(T)} - \frac{1}{2}\xi(T)) \tag{3.13}$$

3.3 Pricing standard call option under stochastic interest rates

In this section we will present the closed-form formula for standard European call and put options under the stochastic interest rates. We give new derivations of these standard results.

Theorem 3.3.1. *Assume that the asset and zero-coupon bond process follow dynamics given in* (3.3)*, under the forward-neutral measure* Q^T*,* A_0

is the initial asset price, K is the strike, T is time to maturity, σ is the volatility for the asset (here and throughout the chapter, we consider only the constant volatility case), ρ is the correlation between Brownian motions of assets and zero-coupon bond.

1. The price C for a European call option at time 0 is

$$C = A_0 \mathcal{N}\left(\frac{\ln(\frac{A_0}{KP(0,T)}) + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) - KP(0,T)\mathcal{N}\left(\frac{\ln(\frac{A_0}{KP(0,T)}) - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) \tag{3.14}$$

2. The price P for a European put option at time 0 is

$$P = KP(0,T)\mathcal{N}\left(\frac{-\ln(\frac{A_0}{KP(0,T)}) + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) - A_0\mathcal{N}\left(\frac{-\ln(\frac{A_0}{KP(0,T)}) - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) \tag{3.15}$$

where $\xi(T)$ can be obtained from (3.8) and it is a known quantity at time 0.

Proof 1. By the Feyman-Kac theorem (or the risk neutral valuation principle), we know that the price of a European call option is written as

$$C = E_{Q^T}[P(0,T)(A_T - K)^+] = P(0,T)E_{Q^T}[(A_T - K)^+] \tag{3.16}$$

note that from (3.13), we actually know that under measure Q^T, conditional on the value of $\xi(T)$, A_T is just a **geometric Brownian motion**. It has distribution

$$A_T \sim LogNormal(\ln(\frac{A_0}{KP(0,T)}) - \frac{1}{2}\xi(T), \xi(T)) \tag{3.17}$$

Recall the following formula: for any random variable $X \sim \mathcal{N}(m, \sigma^2)$,

$$E[e^X 1_{\{e^X \geqslant a\}}] = \exp(m + \frac{\sigma^2}{2})\mathcal{N}(\frac{m + \sigma^2 - \ln(a)}{\sigma}) \tag{3.18}$$

Here we take $m = \ln(\frac{A_0}{KP(0,T)}) - \frac{1}{2}\xi(T)$ and $\sigma^2 = \xi(T)$ then use result (3.18) and (3.16) can be computed by standard ways as in the derivation of the Black-Scholes formula.

2. It is similar to pricing call option, it can be obtained using the parity relation between calls and puts. $\square$

Remark: The formula in **Theorem** 3.3.1 will be reduced to the standard Black-Scholes formula if we have constant interest rates (in this case, just plug in $\sigma_P(u,T) = 0$ and $\rho = 0$ into (3.14), (3.15) and the reader can verify the details.

3.4 Option pricing under point to point transform

Options belonging to this type has payoffs depending on the behavior of the stock price at certain (discrete) time points. It can be priced because we have a one-one correspondence between a time point under the original clock and a time point under the new clock (the new clock can be either deterministic or stochastic). Examples are: standard European vanilla options, forward-start options, cliquet options, chooser options and compound options. This category also includes almost all discretely-monitored exotic options, such as discrete barrier options, discrete Parisian options, etc.

3.4.1 Forward start option under stochastic interest rates

Forward start option for return

Proposition 3.4.1. *Denote $V_1 = \xi(T_1)$ and $V_2 = \xi(T_2)$. This option starts at time T_1 and matures at time T_2, $T_1 < T_2$. The payoff of the option at time T_2 is $\max\left(\frac{S_{T_2}}{S_{T_1}} - k, 0\right)$ for call option, then its price is*

$$C_0 = P(0,T_1)\mathcal{N}(d_1) - kP(0,T_2)\mathcal{N}(d_2) \tag{3.19}$$

where

$$d_1 = \frac{\ln\left(\frac{P(0,T_1)}{kP(0,T_2)}\right) + \frac{1}{2}(V_2 - V_1)}{\sqrt{V_2 - V_1}}, d_2 = d_1 - \sqrt{V_2 - V_1} \tag{3.20}$$

Proof

$$
\begin{aligned}
C_0 &= P\left(0,T_2\right) E^{Q^T}\left[\left(\frac{S_{T_2}}{S_{T_1}}-k\right)^+\right] \\
&= E^{Q^T}\left[\left(P\left(0,T_1\right) e^{B_{V_2-V_1}-\frac{1}{2}(V_2-V_1)} - P\left(0,T_2\right) k\right)^+\right] \\
&= P\left(0,T_1\right) \mathcal{N}(d_1) - kP\left(0,T_2\right) \mathcal{N}(d_2)
\end{aligned}
$$

□

Forward start call option for stock price

Proposition 3.4.2. *The forward start call option for stock price starts at time T_1 and matures at time T_2, $T_1 < T_2$. Its payoff at time T_2 is* $\max\left(S_{T_2} - kS_{T_1}, 0\right)$*, then its price is given as*

$$
C_0 = S_0\mathcal{N}(d_1) - \frac{kS_0 P\left(0,T_2\right)}{P(0,T_1)}\mathcal{N}(d_2) \tag{3.21}
$$

where

$$
d_1 = \frac{\ln\left(\frac{P(0,T_1)}{kP(0,T_2)}\right) + \frac{1}{2}(V_2 - V_1)}{\sqrt{V_2 - V_1}} d_2 = d_1 - \sqrt{V_2 - V_1} \tag{3.22}
$$

Proof

$$
\begin{aligned}
C_0 &= P\left(0,T_2\right) E^{Q^T}\left[\left(S_{T_2} - kS_{T_1}\right)^+\right] \\
&= P\left(0,T_2\right) E^{Q^T}\left[S_{T_1}\left(\frac{S_{T_2}}{S_{T_1}}-k\right)^+\right] \\
&= E^{Q^T}\left[S_{T_1}\right] P\left(0,T_2\right) E^{Q^T}\left[\left(\frac{S_{T_2}}{S_{T_1}}-k\right)^+\right] \\
&= S_0\mathcal{N}(d_1) - \frac{kS_0 P\left(0,T_2\right)}{P(0,T_1)}\mathcal{N}(d_2)
\end{aligned}
$$

The third equality in the proof holds because S_{T_1} and $\frac{S_{T_2}}{S_{T_1}}$ are independent. Note that we can get $P(0,T_1)$ and $P(0,T_2)$ from the market quotes of zero-coupon bond prices with maturity T_1 and T_2, so these two inputs are known at the inception of the contract. □

3.4.2 Chooser option under stochastic interest rates

A chooser option (or *as you like it* option) has the feature that, after a specified period of time, the holder can choose whether the option is a call or a put with the same strike K.

Proposition 3.4.3. *The price of the chooser option is*

$$Price_{chooser} = A_0\left(\mathcal{N}(d_1) - \mathcal{N}(d_4)\right) - KP(0,T_2)\left(\mathcal{N}(d_2) - \mathcal{N}(d_3)\right) \quad (3.23)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{A_0}{KP(0,T_2)}) + \frac{1}{2}\xi(T_2)}{\sqrt{\xi(T_2)}} \\ d_2 &= d_1 - \sqrt{\xi(T_2)} \end{aligned} \quad (3.24)$$

$$\begin{aligned} d_3 &= \frac{\ln(\frac{KP(0,T_2)}{A_0}) + \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}} \\ d_4 &= d_3 - \sqrt{\xi(T_1)} \end{aligned} \quad (3.25)$$

Proof Suppose that the choice is made at time T_1, and T_2 is the maturity of the option, where $T_2 > T_1$. Denote $V_1 = \xi(T_1)$ and $V_2 = \xi(T_2)$ for notational convenience. Then the payoff of the chooser option at time T_1 is $\max(c, p)$, by the put-call parity, we have

$$\begin{aligned} \max(C, P) &= \max(C, C + KP(T_1, T_2) - A_{T_1}) \\ &= C + \max(KP(T_1, T_2) - A_{T_1}, 0) \end{aligned} \quad (3.26)$$

Thus we can see that the chooser option is actually a package of "a call option with strike K and maturity T_2" and "a put option with strike $KP(T_1, T_2)$ and maturity T_1".

For the call part,

$$C = A_0\mathcal{N}\left(\frac{\ln(\frac{A_0}{KP(0,T_2)}) + \frac{1}{2}\xi(T_2)}{\sqrt{\xi(T_2)}}\right) - KP(0,T_2)\mathcal{N}\left(\frac{\ln(\frac{A_0}{KP(0,T_2)}) - \frac{1}{2}\xi(T_2)}{\sqrt{\xi(T_2)}}\right) \quad (3.27)$$

For the put part,

$$\begin{aligned}
P &= P(0,T_1)E^{Q^T}\left[(KP(T_1,T_2)-A_{T_1})^+\right] \\
&= P(0,T_1)E^{Q^T}\left[P(T_1,T_2)\left(K-\frac{A_{T_1}}{P(T_1,T_2)}\right)^+\right]
\end{aligned} \tag{3.28}$$

Note that from (3.12) we know that

$$\frac{A_{T_1}}{P(T_1,T_2)} = \frac{A_0}{P(0,T_2)}\exp(B_{\xi(T_1)} - \frac{1}{2}\xi(T_1)) \tag{3.29}$$

because $P(0,T_2)$ is a constant obtained from risk-free coupon quotes from market, we can see that $\frac{A_{T_1}}{P(T_1,T_2)}$ only depends on information up to T_1 and does **not** depend on any information in $[T_1,T_2]$, so it's **independent** of $P(T_1,T_2)$. Then we have

$$\begin{aligned}
P &= P(0,T_1)E^{Q^T}\left[P(T_1,T_2)\left(K-\frac{A_{T_1}}{P(T_1,T_2)}\right)^+\right] \\
&= P(0,T_1)E^{Q^T}\left[P(T_1,T_2)\right]E^{Q^T}\left[\left(K-\frac{A_{T_1}}{P(T_1,T_2)}\right)^+\right]
\end{aligned} \tag{3.30}$$

We first calculate

$$\begin{aligned}
&E^{Q^T}\left[\left(K-\frac{A_{T_1}}{P(T_1,T_2)}\right)^+\right] \\
&= K\mathcal{N}\left(\frac{\ln(\frac{KP(0,T_2)}{A_0}) + \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}}\right) - \frac{A_0}{P(0,T_2)}\mathcal{N}\left(\frac{\ln(\frac{KP(0,T_2)}{A_0}) - \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}}\right)
\end{aligned}$$

Denote

$$\psi(T) = \int_0^T (\sigma_P(u,T)-\sigma_P(u,t))^2\,du$$

then similarly by Dubins-Schwartz theorem, we have

$$\begin{aligned}
E^{Q^T}\left[P(T_1,T_2)\right] &= E^{Q^T}\left[\frac{P(0,T_2)}{P(0,T_1)}e^{-B_{\psi(T_1)}+\frac{1}{2}\psi(T_1)}\right] \\
&= \frac{P(0,T_2)}{P(0,T_1)}
\end{aligned} \tag{3.31}$$

So now we finally have

$$P = P(0,T_1)E^{Q^T}\left[P(T_1,T_2)\right]E^{Q^T}\left[\left(K - \frac{A_{T_1}}{P(T_1,T_2)}\right)^+\right]$$
$$= KP(0,T_2)\mathcal{N}\left(\frac{\ln(\frac{KP(0,T_2)}{A_0}) + \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}}\right) - A_0\mathcal{N}\left(\frac{ln(\frac{KP(0,T_2)}{A_0}) - \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}}\right) \tag{3.32}$$

and

$$\begin{aligned} Price_{chooser} &= C + P \\ &= A_0\mathcal{N}\left(\frac{\ln(\frac{A_0}{KP(0,T_2)}) + \frac{1}{2}\xi(T_2)}{\sqrt{\xi(T_2)}}\right) - KP(0,T_2)\mathcal{N}\left(\frac{ln(\frac{A_0}{KP(0,T_2)}) - \frac{1}{2}\xi(T_2)}{\sqrt{\xi(T_2)}}\right) \\ &+ KP(0,T_2)\mathcal{N}\left(\frac{\ln(\frac{KP(0,T_2)}{A_0}) + \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}}\right) - A_0\mathcal{N}\left(\frac{\ln(\frac{KP(0,T_2)}{A_0}) - \frac{1}{2}\xi(T_1)}{\sqrt{\xi(T_1)}}\right) \\ &= A_0\left(\mathcal{N}(d_1) - \mathcal{N}(d_4)\right) - KP(0,T_2)\left(\mathcal{N}(d_2) - \mathcal{N}(d_3)\right) \end{aligned}$$

□

3.4.3 Compound option under stochastic interest rates

A compound option is simply "an option on an option". There are four types of compound options and here we just consider one of them the call on a call. The other types can be dealt with similarly.

At time T_1, this compound option offers the holder the right to enter into a second call option or receive the strike price K_1. This second call option has strike $K_2 = \frac{L}{P(T_1,T_2)}$ and expires at time T_2, where L is a constant. Note that here the second strike is an *accumulated* strike and we can show that in this case we can arrive at a closed form solution for compound option under stochastic interest rates. As shown in Frey and Sommer (1998), for K_2 constant, the Geske (1979) formula can not be readily extended and it is not a trivial problem. We leave it for future research.

Proposition 3.4.4. *The price of this Compound option under stochastic* *interest rates with the second strike being the "accumulated" strike is*

$$C(A_0, 0) = A_0\mathcal{N}_2\left(a_1, b_1, \rho\right) - LP(0, T_1)\mathcal{N}_2\left(a_2, b_2, \rho\right) - K_1 P(0, T_1)\mathcal{N}(a_2) \tag{3.33}$$

where

$$\begin{aligned} a_2 &= \frac{\ln\left(\frac{A_0}{\tilde{A}_{T_1} P(0,T_1)}\right) - \frac{1}{2}V_1}{\sqrt{V_1}} \\ b_2 &= \frac{\ln\left(\frac{A_0}{LP(0,T_1)}\right) - \frac{1}{2}V_2}{V_2} \\ a_1 = a_2 + \sqrt{V_1}&; b_1 = b_2 + \sqrt{V_2} \end{aligned} \tag{3.34}$$

Proof Let $C(A_0, 0)$ denote the value of this compound call on call option at current time and let $C(\tilde{A}_{T_1}, T_1)$ denote the value of the underlying call option at time T_1. Note that the compound option will be exercised at T_1 only when $C(\tilde{A}_{T_1}, T_1) > K_1$. For notational convenience, we denote $V_1 = \xi(T_1), V_2 = \xi(T_2), V_{12} = \xi(T_2) - \xi(T_1)$. We first calculate the price of the second call option at time T_1

$$\begin{aligned} \tilde{C}(A_{T_1}, T_1) &= E^{Q^T}\left[P(T_1, T_2)\max\left(A_{T_2} - K_2, 0\right)\right] \\ &= E^{Q^T}\left[P(T_1, T_2)\max\left(\frac{A_0}{P(T_1, T_2)} e^{B_{V_{12}} - \frac{1}{2}V_{12}} - K_2, 0\right)\right] \\ &= E^{Q^T}\left[\max\left(A_0 e^{B_{V_{12}} - \frac{1}{2}V_{12}} - K_2 P(T_1, T_2), 0\right)\right] \\ &= E^{Q^T}\left[\max\left(A_0 e^{B_{V_{12}} - \frac{1}{2}V_{12}} - L, 0\right)\right] \\ &= A_{T_1}\mathcal{N}(d_1) - L\mathcal{N}(d_2) \end{aligned} \tag{3.35}$$

where

$$d_1 = \frac{\ln\left(\frac{L}{A_{T_1}}\right) + \frac{1}{2}V_{12}}{\sqrt{V_{12}}}; d_2 = d_1 - \sqrt{V_{12}} \tag{3.36}$$

Here the first equality is because we have the representation of the asset price given in equation (3.13) and we just set the starting time to T_1. We can easily see that in this case, the price of the call option is monotone with the stock price, thus if we set $\tilde{C}(A_{T_1}, T_1) = K_1$, we can solve for the unique critical value $\tilde{A}_{T_1}$ for A_{T_1} that satisfies the above equation. It is obtained by solving the nonlinear equation

$$\tilde{C}(\tilde{A}_{T_1}, T_1) = K_1 \tag{3.37}$$

After we get the unique value for $\tilde{C}(A_{T_1}, T_1)$, we can calculate the price of this compound option at time 0

$$\begin{aligned} C(A_0, 0) &= P(0, T_1) E^{Q^T} \left[\max \left(\tilde{C}(A_{T_1}, T_1) - K_1, 0 \right) \right] \\ &= P(0, T_1) \int_0^{\infty} \max \left(\tilde{C}(A_{T_1}, T_1) - K_1, 0 \right) \psi \left(A_{T_1}; A_0 \right) dA_{T_1} \\ &= P(0, T_1) \int_{\tilde{A}_{T_1}}^{\infty} \left[A_{T_1} \mathcal{N}(d_1) - L \mathcal{N}(d_2) - K_1 \right] \psi \left(A_{T_1}; A_0 \right) dA_{T_1} \end{aligned} \tag{3.38}$$

where Q^T is the forward-neutral measure and the transition density function $\psi \left(A_{T_1}; A_0 \right)$ under this measure is given by

$$\psi \left(A_{T_1}; A_0 \right) = \frac{1}{A_{T_1} \sqrt{2\pi \xi(T_1)}} \exp \left(- \frac{\left[\ln \left(\frac{A_{T_1}}{A_0} \right) + \frac{1}{2} \xi(T_1) + \ln(P(0, T_1)) \right]^2}{2\xi(T_1)} \right) \tag{3.39}$$

We calculate equation (3.38) term by term.

$$\begin{aligned} & - K_1 P(0, T_1) \int_{\tilde{A}_{T_1}}^{\infty} \psi \left(A_{T_1}; A_0 \right) dA_{T_1} \\ &= -K_1 P(0, T_1) E^{Q^T} \left[\mathbb{1}_{\{A_{T_1} \leqslant \tilde{A}_{T_1}\}} \right] \\ &= -K_1 P(0, T_1) Q^T \left(A_{T_1} \geqslant \tilde{A}_{T_1} \right) \\ &= -K_1 P(0, T_1) \mathcal{N}(a_2) \end{aligned} \tag{3.40}$$

where

$$a_2 = \frac{\ln \left(\frac{A_0}{\tilde{A}_{T_1} P(0, T_1)} \right) - \frac{1}{2} V_1}{\sqrt{V_1}} \tag{3.41}$$

similarly,

$$
\begin{aligned}
& - LP(0,T_1)\int_{\tilde{A}_{T_1}}^{\infty}\mathcal{N}(d_2)\psi\left(A_{T_1};A_0\right)dA_{T_1} \\
& = -LP(0,T_1)E^{Q^T}\left[\mathbb{1}_{\{A_{T_1}\geqslant \tilde{A}_{T_1}\}}\mathbb{1}_{\{A_{T_2}\geqslant \frac{L}{P(T_1,T_2)}\}}\right] \\
& = -LP(0,T_1)Q^T\left[B_{V_1}\geqslant \ln\left(\frac{\tilde{A}_{T_1}P(0,T_1)}{A_0}\right)+\frac{1}{2}V_1; B_{V_2}\geqslant \ln\left(\frac{LP(0,T_1)}{A_0}\right)+\frac{1}{2}V_2\right] \\
& = -LP(0,T_1)\mathcal{N}_2\left(a_2,b_2,\rho\right) \qquad (3.42)
\end{aligned}
$$

where

$$
b_2 = \frac{\ln\left(\frac{A_0}{LP(0,T_1)}\right)-\frac{1}{2}V_2}{V_2} \qquad (3.43)
$$

and $\mathcal{N}_2\left(...\right)$ is the bivariate normal distribution. For the last equality in equation (3.42), note that $COV(B_{V_1},B_{V_2}) = V_1$, thus the correlation between these two Brownian motions is $\rho = \frac{COV(B_{V_1},B_{V_2})}{\sqrt{VAR(B_{V_1})VAR(B_{V_2})}} = \sqrt{\frac{V_1}{V_2}}$ Similarly,

$$
P(0,T_1)\int_{\tilde{A}_{T_1}}^{\infty}A_{T_1}\mathcal{N}(d_1)\psi\left(A_{T_1};A_0\right)dA_{T_1} = A_0\mathcal{N}_2\left(a_1,b_1,\rho\right) \qquad (3.44)
$$

where

$$
a_1 = a_2+\sqrt{V_1}; b_1 = b_2+\sqrt{V_2} \qquad (3.45)
$$

Then we arrive at equation (3.33) if we combine the results (3.40), (3.42) and (3.44). Here we omit some tedious calculations and similar derivations can be found in Kwok (2008). □

3.5 Conclusion of Chapter 3

In this chapter we have used the “deterministic time-change idea” to price various exotic European options under stochastic interest rates.

However, the time changes we discuss are not invariant under interval transform. For example, assume the (deterministic) time change is given by $\xi(T) = \int_0^T f(s)ds$, then for a time interval under the original clock $[t_1,t_2]$, we have correspondingly a new time interval under the new stochastic clock

$[\xi(t_1), \xi(t_2)]$. So if the option payoff involves the length of the interval, then we can not use our time change technique to price it *exactly*. That is because usually we don't have $\xi(t_2 - t_1) = \xi(t_2) - \xi(t_1)$. One example will be the pricing of the standard Parisian option under stochastic interest rates. Although we can not get the exact price of it under the time-change method, we can find either a lower bound or an upper bound for its price. Details will be discussed in the Chapter 4.

Chapter 4

Parisian option under Stochastic interest rates

The study of exotic options in the context of stochastic interest rates is a rather difficult problem. It is usually solved in the financial industry by means of Monte-Carlo simulations or partial differential equations. In this chapter we provide both an upper bound and a lower bound for the price of a Parisian option when the interest rate is stochastic. Parisian options extend barrier options in that the activation and deactivation depend not only on the fact that the underlying process hits a given level, but also depend on the time spent by the underlying process beyond this level.

We consider the case when the option's barrier is stochastic similarly as Briys and de Varenne (1997). The barrier level at time u prior to the maturity T can be interpreted as the principal L of the debt of a company with maturity T, discounted at the spot rate at time u, i.e. the barrier threshold at time u is

$$B(u) := LP(u, T),$$

where $P(u, T)$ is the market value at time u of a riskless zero-coupon bond with maturity T. This chapter is not only a technical contribution to the pricing of Parisian options, but also a contribution to the literature on structural models. The simplest structural model of the bankruptcy of a company was proposed by Merton (1974). Let us briefly recall this theory. Consider a simple company starting at 0, endowed with equity E_0, issuing initially the amount D_0 of zero-coupon bonds maturing at T and promising L. E_0 and D_0 are invested in the lognormally distributed assets A_0. Following Merton, the equity is a call option on the assets of the firm, with strike price the principal L of the debt, and maturity T of the debt. The debt is a risky zero-coupon bond, it is the sum of a *long* position in a risk-free zero-coupon bond (with same maturity and principal) and a *short* position in a ("default") put on the assets of the firm, with strike L and maturity T. In the case when the interest rate r is constant, the Black-Scholes formulas can be used in this context. Firstly, the Merton approach can be extended to the case of stochastic interest rates very easily when the firm issues such a simple debt profile (in the formulas, one simply needs to discount using the riskfree zero-coupon price $P(0, T)$ instead of the factor e^{-rT}). Secondly, in the Merton approach, default can occur only at the maturity of the risky zero-coupon bonds. Indeed, in the situation where the assets process A_t dives down between 0 and T, it can easily be conceived that bankruptcy will be declared before the maturity of the debt. The main contribution of Black and Cox (1976) is to include the possibility of an early default prior to maturity: when the assets of the firm hit an *ad hoc* barrier, the firm defaults. This barrier can be of any kind; very often it corresponds to the principal of the debt. In this framework, the debt becomes a path-dependent exotic option on the assets. Formulas derived by Black and Cox (1976) are obtained in a Black and Scholes setting when interest rates are constant. Longstaff and Schwartz (1995) (corrected by Collin-Dufresne and Golstein (2001)) extend the work of Black and Cox (1976) to the case when interest rates are stochastic, and when the barrier level that triggers bankruptcy is constant and deterministic. Briys and de Varenne (1997) define the default barrier as a fixed quantity discounted at the spot (riskless) rate up to the maturity of the debt. As soon as the barrier is crossed, bondholders receive an exogenous fraction of the remaining assets. In his model, interest rates are stochastic and therefore the barrier level is also stochastic.

The contributions described previously allow us to model very well the case when the default happens according to the Chapter 7 of the US bankruptcy code. This means that when assets are too low to meet liabilities, the company is liquidated and bondholders are reimbursed. This is however not the case when the bankruptcy procedure follows the Chapter 11 of the US bankruptcy code[1]. Francois and Morellec (2004) have already explained how to model the Chapter 11 using Parisian options. Their work is done under the Black and Scholes market assumptions. The value of the risky debt can now be modeled as a Parisian option written on the assets of the company. This chapter can be seen as an extension of their approach

[1]The US bankruptcy code distinguishes between Chapter 7 and Chapter 11 bankruptcy procedures. According to Chapter 7 bankruptcy procedure, the default and the liquidation dates coincide. In contrast, Chapter 11 bankruptcy procedure describes a more realistic procedure, in the sense that default and liquidation are distinguishable events. Similar bankruptcy procedures can be found also in France, Germany and Japan, etc.

to the case when interest rates are stochastic, and an extension of Briys and de Varenne (1997) when the company is liquidated according to the Chapter 11 instead of the Chapter 7.

To derive the price of the Parisian option, we make use of the deterministic time change idea extensively. We note that conditional on the deterministic time change, the price of a Parisian option under stochastic interest rates lies between two bounds that are respectively equal to the prices of standard Parisian options in the Black Scholes framework with different transformed parameters. We then show that these upper and lower bounds are very close to the true price by further exploring the property of the deterministic time change $\xi(t)$ for $t \in [0, T]$. In practice these bounds are then very close to each other and therefore provide an accurate approximation of the price of a Parisian option under stochastic interest rates. This chapter directly extends the literature on the pricing of Parisian options as well as the literature on the pricing of exotic options in a stochastic term structure of interest rates (See Bernard et al. (2008)). Parisian options have been studied by Chesney, Jeanblanc and Yor (1997). A survey of the formulas in the Black and Scholes model can be found in Labart and Lelong (2009).

4.1 Setting

We motivate the problem of pricing a Parisian option in a stochastic interest rate environment by evaluating the risky debt of a company that is regulated by the Chapter 11 of the US bankruptcy law.

4.1.1 Parisian Option

In this paper, we study the down Parisian options: this is not a restriction since there are parity relationships between the different Parisian options in the same way as for barrier options (see Labart and Lelong (2009)). Denote by A_t the assets of the company at time t. The assets' level is monitored continuously. We observe the time spent *below* the barrier $LP(t, T)$. To do so, we introduce g_t^L which is the last time before t the process A_s reaches the value $LP(s,T)$. It is defined by

$$g_t^L = \sup\{s \leqslant t \mid A_s = LP(s,T)\} \tag{4.1}$$

Define then $G_{D,L}(A)$, as the first time the underlying A has remained more than time D below the barrier L, by

$$G_{D,L} = G_{D,L}(A) = \inf\{t > 0 \mid (t - g_t^L)\mathbb{1}_{A_t \leqslant LP(t,T)} \geqslant D\} \tag{4.2}$$

Consider an arbitrage-free and complete market equipped with the risk neutral probability measure Q. We are studying the value of a risky bond with face value L paid at time T when the assets are monitored continuously by the regulators. The company is liquidated when the assets continuously spent more than D below the level of the barrier $LP(t,T)$. The price of the risky bond can then be expressed as

$$p_0 = E_Q\left[e^{-\int_0^T r_s ds}\min(A_T, L)\mathbb{1}_{G_{D,L}>T}\right] \tag{4.3}$$

where r_s is the instantaneous riskfree interest rate at time s.

This can be interpreted as follows. If the Parisian time $G_{D,L}$ occurs before the maturity then the company is liquidated and we first assume that there is no recovery for the bondholders. The remaining assets are used to pay for the liquidation costs. The bondholders receive some cashflows if the company is still alive at time T, and they will receive the minimum between L and A_T, we then allow for default at maturity if the assets are not enough to pay the principal L.

For a more realistic setting, one needs to add additional costs at maturity. It would be also interesting to allow for the case when default happens prior to T but that bondholders also receive a percentage α of the assets at time of default. In this case, the initial price of the debt would be given by

$$d_0 = E_Q\left[e^{-\int_0^{G_{D,L}} r_s ds}\alpha A_{G_{D,L}}\mathbb{1}_{G_{D,L}<T}\right] + E_Q\left[e^{-\int_0^T r_s ds}\min\left(\alpha A_T, L\right)\mathbb{1}_{G_{D,L}>T}\right] \tag{4.4}$$

In this chapter we describe more generally how to calculate a lower and an upper bound for the following covenant

$$E_Q\left[e^{-\int_0^T r_s ds}\min(A_T, L)\mathbb{1}_{G_{D,L}>T}\right] \tag{4.5}$$

which corresponds to any European option with a Parisian activation condition. This includes the case of down and in call and put Parisian options as special cases. When the interest rate is constant, formulas can be obtained following the approach of Chesney, Jeanblanc and Yor (1997).

4.1.2 Methodology

Our goal is to provide a lower bound and an upper bound to the expression (4.5) when interest rates are stochastic. The first step is to change the measure from the risk neutral probability Q to the forward neutral probability measure Q_T to get rid of the stochastic discount factor $e^{-\int_0^T r_s ds}$. Under Q_T, the price (4.5) becomes

$$P(0,T)E_{Q_T}\left[\min(A_T, L)\mathbb{1}_{G_{D,L}>T}\right]. \tag{4.6}$$

The stochastic discount factor has then disappeared. Another change of measure will be useful to reduce the problem to a simpler problem where the Parisian condition for the assets becomes a Parisian condition for a standard geometric Brownian motion. We will then make use of the pricing formulae existing in the Black and Scholes framework.

To summarize, the derivation of the bounds of the price of a Parisian option is organized as follows. We first calculate the dynamics for the discounted asset price under the forward neutral measure Q_T. This expression involves a martingale. Using the Dubins-Schwarz theorem, we represent this martingale as a time-changed Brownian motion. We then further simplify the problem using a second change of measure from Q_T to $\widetilde{Q}$ and obtain an expression for the bounds of the price of a Parisian option in a stochastic interest rates environment. Finally we reduce the calculations of the bounds to special cases of the Laplace transform formulas given in Labart and Lelong (2009).

4.2 Asset and interest rate dynamics

We assume that the assets of the company A_t follows a log-normal dynamic correlated to the interest rates, which themselves possess an exponential volatility structure σ_P. This corresponds to a standard Hull and White specification. The interest rate model considered here is driven by a unique factor, correlated to the one of the asset, as mentioned before.

4.2.1 Under the risk neutral probability

Setting $\nu > 0$ and $a > 0$, the volatility structure expresses simply as $\sigma_P(t,T) = \frac{\nu}{a}\left(1 - e^{-a(T-t)}\right)$. Under the risk-neutral measure Q, the dynamics of the assets A_t and of the zero-coupon bond $P(t,T)$ are respectively given as

$$\frac{dA_t}{A_t} = r_t dt + \sigma dZ^{\mathrm{Q}}(t) \tag{4.7}$$

and

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T) dZ_1^{\mathrm{Q}}(t) \tag{4.8}$$

where $Z^{\mathrm{Q}}(t)$ and $Z_1^{\mathrm{Q}}(t)$ are standard Q-Brownian motions with correlation coefficient equal to ρ.

Let us now construct a Brownian motion Z_2^{Q} independent from Z_1^{Q}. It is possible to split up Z^{Q} into the two following components by Cholesky decomposition

$$dZ^{\mathrm{Q}}(t) = \rho dZ_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathrm{Q}}(t)$$

We have therefore decorrelated the pure interest rate risk from the other sources of risk. The dynamics of the assets given in (4.7) can now be reexpressed as

$$\frac{dA_t}{A_t} = r_t dt + \sigma \left(\rho dZ_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathrm{Q}}(t)\right)$$

4.2.2 Under the forward neutral probability

Recall that the Radon-Nikodym density allows us to build the forward-neutral measure Q_T

$$\frac{d\mathrm{Q}_T}{dQ} = e^{-\int_0^T \sigma_P(s,T)dZ_1^{\mathrm{Q}}(s) - \frac{1}{2}\int_0^T \sigma_P^2(s,T)ds}$$

In this case, the short-term interest rate dynamics obey the following relationship

$$dr_t = a(\theta_t - r_t)dt + \nu dZ_1^{\mathrm{Q}_T}(t)$$

where $\theta_t = \theta^* - \frac{\nu^2}{a^2}\left(1 - e^{-a(T-t)}\right)$ and where we have defined a new Brownian motion $Z_1^{\mathrm{Q}_T}$ satisfying under Q_T the relationship $dZ_1^{\mathrm{Q}_T} = dZ_1^{\mathrm{Q}} + \sigma_P(t,T)dt$ by the Girsanov Theorem.

We also define $Z_2^{\mathrm{Q}_T}$ such that $Z_1^{\mathrm{Q}_T}$ and $Z_2^{\mathrm{Q}_T}$ are non-correlated Q_T-Brownian motions. The dynamics of A_t and $P(t,T)$ under Q_T finally write as

$$\frac{dA_t}{A_t} = (r_t - \sigma\rho\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{\mathrm{Q}_T} + \sqrt{1-\rho^2}dZ_2^{\mathrm{Q}_T}\right)$$

and

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{\mathrm{Q}_T}$$

Upon integration of these two dynamics, one obtains

$$\begin{aligned} A_t = \frac{A_0}{P(0,t)} \exp\Bigg(&\int_0^t (\sigma_P(u,t) + \rho\sigma\ )dZ_1^{\mathrm{Q}_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\ dZ_2^{\mathrm{Q}_T}(u) \\ &+ \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2}\right) du\Bigg) \end{aligned}$$

and

$$\begin{aligned} P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\Bigg(&-\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))\, dZ_1^{\mathrm{Q}_T} \\ &+\frac{1}{2}\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))^2\, du\Bigg) \end{aligned}$$

Finally, note that the following dynamic will also be useful in the coming developments.

$$\frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} \exp\left(\int_0^t (\sigma_P(u,T)+\rho\sigma)\, dZ_1^{\mathrm{Q}_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2} dZ_2^{\mathrm{Q}_T}(u) \right.$$
$$\left. -\frac{1}{2}\int_0^t \left((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2)\right) du\right) \quad (4.9)$$

The process of the discounted assets indeed plays an important role. The intuition can easily be seen from (4.1) and (4.2), which can be rewritten as

$$g_t^L = \sup\left\{s \leqslant t \mid \frac{A_s}{P(s,T)} = L\right\} \quad (4.10)$$

and

$$G_{D,L}(S) = \inf\{t > 0 \mid (t - g_t^L)\mathbb{1}_{\frac{A_t}{P(t,T)} \leqslant L} \geqslant D\}. \quad (4.11)$$

Both of the above random times are linked to the process of the discounted assets.

4.3 Derivations

This section contains the different steps of the proof to obtain lower and upper bounds of prices of European Parisian options in a stochastic interest rate environment.

4.3.1 Change of measure from Q to Q_T

Now we denote for $t \in [0,T]$,

$$N_t = \int_0^t (\sigma_P(u,T)+\rho\sigma)\, dZ_1^{\mathrm{Q}_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2} dZ_2^{\mathrm{Q}_T}(u)\,. \quad (4.12)$$

The quadratic variation of the above martingale can be easily calculated. It is equal to

$$\xi(t) = \int_0^t \left((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2)\right) du \quad (4.13)$$

for $t \in [0,T]$. Under the assumption that the volatility structure of the short rate is exponential, one has

$$\sigma_P(t,T) = \frac{\nu}{a}\ \left(1 - e^{-a(T-t)}\right), t \in [0,T].$$

Replacing the above expression of σ_P into equation (4.13), we can calculate an explicit expression for $\xi(t)$

$$\begin{aligned} \xi(t) &= \frac{v^2}{2a^3}e^{-2aT}e^{2at} - \frac{2v}{a^2}e^{-aT}\left(\frac{v}{a} + \rho\sigma\right)e^{at} \\ &+ \left[\sigma^2 + \left(2\rho\sigma + \frac{v}{a}\right)\frac{v}{a}\right]t + \frac{2v}{a^2}e^{-aT}\left(\frac{v}{a} + \rho\sigma\right) - \frac{v^2}{2a^3}e^{-2aT} \end{aligned} \quad (4.14)$$

So from this equation (4.14), we observe that

$$\lim_{t\to\infty} \xi(t) = +\infty \quad (4.15)$$

Let us study the monotonicity of $\xi(T)$. We start by calculating the first order derivative of $\xi(t)$

$$\xi'(t) = \frac{v^2}{a^2}e^{-2aT}e^{2at} - \frac{2v}{a}e^{-aT}(\frac{v}{a} + \rho\sigma)e^{at} + \sigma^2 + (2\rho\sigma + \frac{v}{a})\frac{v}{a}$$

which can also be written as

$$\xi'(t) = \frac{v^2}{a^2}e^{-2aT}\left[e^{at} - (1 + \rho\sigma\frac{a}{v})e^{aT}\right]^2 + (1-\rho^2)\sigma^2 > 0. \quad (4.16)$$

This means that the function $\xi(t)$ is always monotonically increasing. From (4.13) we know that $\xi(0) = 0$. Then $\xi(t)$ is therefore non negative and is a bijection of the interval $[0,T]$ to the interval $[0, \xi(T)]$. The second order derivative of $\xi(t)$ is given by

$$\xi''(t) = \frac{2v^2}{a}e^{-2aT}\left[e^{at} - \frac{1}{2}\left(1 + \rho\sigma\frac{a}{v}\right)e^{aT}\right]^2 - \frac{v^2}{2a}\left(1 + \rho\sigma\frac{a}{v}\right)^2 \quad (4.17)$$

Further properties of the time-change function $\xi(t)$ is given in the Section 4.4. Especially we show in that section that $\xi(t)$ is a **concave** function for $t \in [0,T]$.

Recall that $\xi(t)$ is the quadratic variation of N_t (expressions (4.12) and (4.13)). Therefore since $\xi(t)$ also satisfies (4.15), the assumptions of the Dubin-Schwarz Theorem of Time Change of Martingales are verified (see Karatzas and Shreve (1991), p174, Thm 3.4.6), and there exists a Q^T-Brownian Motion B such that

$$\forall t \in [0,T], \quad N_t = B_{\xi(t)}. \tag{4.18}$$

Under Q_T, one has

$$\frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} e^{N_t - \frac{1}{2}\xi(t)} = \frac{A_0}{P(0,T)} e^{B_{\xi(t)} - \frac{1}{2}\xi(t)}. \tag{4.19}$$

We can further simplify the expression of the discounted assets by getting rid of the drift of the Brownian motion involved in the exponential term. To do so, we proceed with a second change of measure.

4.3.2 Change of measure from Q_T to $\widetilde{Q}$

Now we intend to change the probability measure from the forward-neutral measure Q^T to another measure $\widetilde{Q}$. Since $\xi(t)$ is a bijection, one has $\forall t \in [0,T], s = \xi(t) \in [0, \xi(T)], t = \xi^{-1}(s)$. Using the one-dimensional Girsanov Theorem (see Shreve (2004), p212, theorem 5.1.4), let $m(u) = -\frac{1}{2}$, the assumptions of the theorem are satisfied, and we can define

$$\widetilde{B_s} = B_s - \frac{1}{2}s, s \in [0, \xi(T)] \tag{4.20}$$

and the Radon-Nikodym derivative is

$$\frac{dQ^T}{d\widetilde{Q}} \mid F_T = e^{-\frac{1}{2}\widetilde{B}_{\xi(T)} - \frac{1}{8}\xi(T)} \tag{4.21}$$

Here $\widetilde{B_s}$ is a standard Brownian Motion under the new measure $\widetilde{Q}$. Under $\widetilde{Q}$, the discounted asset process is simply a geometric Brownian motion

$$\frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} e^{\widetilde{B}_{\xi(t)}} \tag{4.22}$$

4.3.3 Parisian Time

The Parisian time refers to the time $G_{D,L}$ defined in (4.2). The deterministic change of time we have employed has an effect on the Parisian time. The Parisian option is indeed linked to the time spent by the original process below a given level. It is not obvious how it relates to the time spent by the new Brownian motion after the change of time ξ. We now discuss this issue and how to obtain bounds of the Parisian time and thus also the price of the Parisian option.

Barrier time (First passage time) One needs to keep in mind that we are working under the new measure $\widetilde{Q}$. First we discuss the event that the asset price hits the discounted barrier given by $LP(u,T)$. This is a helpful step to understand how the change of time affects the first hitting times and then how it affects the Parisian times. We have

$$\begin{aligned}\{A_u = LP(u,T)\} &= \left\{\frac{A_u}{P(u,T)} = L\right\} \\ &= \left\{\frac{A_0}{P(0,T)} e^{\widetilde{B}_{\xi(u)}} = L\right\} \\ &= \left\{\widetilde{B}_{\xi(u)} = \ln\left(\frac{LP(0,T)}{A_0}\right)\right\} \\ &= \{\widetilde{B}_{\xi(u)} = b\}\end{aligned} \tag{4.23}$$

where $b = \ln\left(\frac{LP(0,T)}{A_0}\right)$. There is therefore a bijection between the first hitting time of $\widetilde{B}$ to the constant b with the first hitting time of the assets with the discounted stochastic barrier. However this is not the case for the Parisian times.

For each random time, we can define it in two ways. To avoid the confusion, we will mention the underlying process. If we refer to $\widetilde{B}$, it means that we are working under $\widetilde{Q}$ and that the range of time is $[0, \xi(T)]$. In the absence of a reference to $\widetilde{B}$, it means that we refer to the original process, and the real time in $[0, T]$. Let us illustrate this in the case of the first hitting time to the given level $b > 0$, we have

$$t_b(\widetilde{B}) = \inf\{s > 0, \quad \widetilde{B}_s = b\} \in [0, \xi(T)] \tag{4.24}$$

note the expression (4.23) which gives the equivalence of events

$$t_b = \inf\{t > 0, \quad A_t = LP(t,T)\} = \xi^{-1}(t_b(\widetilde{B})) \in [0,T] \tag{4.25}$$

Proposition 4.3.1 (Price of a Barrier Option with Stochastic Interest Rates)**.** *Let us denote by $DOB(L,T)$, the price of a down and out Barrier option with payoff*

$$L\mathbb{1}_{t_L > T}$$

where L is the level of the barrier, T the maturity of the option, t_L is the first hitting time of the process S to the level L, and where $S_0 > L$. The underlying process evolves as $dS_t = rS_t dt + \sigma S_t dW_t$ under Q and the interest rate is constant (Black and Scholes framework).

The price of the Barrier option with stochastic interest rates with payoff

$$L\mathbb{1}_{T_L > T}$$

where T_L is the first hitting time of the assets A_t to the level $LP(t,T)$ is given by

$$P(0,T)\Phi\left(\frac{\ln\left(\frac{A_0}{LP(0,T)}\right) - \frac{\xi(T)}{2}}{\sqrt{\xi(T)}}\right) + \frac{A_0}{L}\Phi\left(\frac{\ln\left(\frac{A_0}{LP(0,T)}\right) + \frac{\xi(T)}{2}}{\sqrt{\xi(T)}}\right)$$

where $\tau(T) = \int_0^T (\sigma_P(u,t) + \rho\sigma)^2 + \sigma^2(1-\rho^2)du$ and Φ is the cdf of a normal distribution $\mathcal{N}(0,1)$.

Proof. The proof can be found in Bernard et al (2008), expression E_1 in Proposition 3.1 of that paper. □

Parisian time The problem is very different for the Parisian times and we do not necessarily have a bijection between them. Consider the first time that the Brownian motion $\widetilde{B}$ makes an excursion longer than some time D below the level b. Let us denote it by $G_{D,b}(\widetilde{B})$. We want to understand how it relates to $G_{D,L}$ defined in (4.2). To define $G_{D,b}(\widetilde{B})$ properly, we need to first define the following

$$g_s^b(\widetilde{B}) = \sup\{u \leqslant s, \quad \widetilde{B}_u = b\} \in [0, \xi(T)]$$

Recall that

$$G_{D,L} = \inf \left\{ t > 0,\ (t - g_t^L) \mathbb{1}_{\{A_t < LP(t,T)\}} \geqslant D \right\} \in [0, T]$$

Similarly, we can define this Parisian time directly for the process $\widetilde{B}$,

$$G_{D,b}(\widetilde{B}) = \inf \left\{ s > 0,\ (s - g_s^b(\widetilde{B})) \mathbb{1}_{\{\widetilde{B}_s < b\}} \geqslant \xi(D) \right\} \in [0, \xi(T)]$$

However, the problem with the Parisian times is much more difficult because unfortunately $G_{D,L}$ and $G_{D,b}(\widetilde{B})$ cannot be related directly using ξ^{-1} as it was possible in (4.25) for the case of first hitting times.

To find a relationship, we would need to relate the condition $(s - g_s^b(\widetilde{B})) \geqslant \xi(D)$ to the condition $(t - g_t^L) \geqslant D$, where $s = \xi(t) \in [0, \xi(T)]$, and $t \in [0, T]$. An original excursion of length D relates to an excursion of length $\xi(t) - \xi(t - D)$ after the time change (and not directly to $\xi(D)$). The problem is that in general this quantity $\xi(t) - \xi(t - D)$ depends on t. If it is independent of t, then it is equal to $\xi(D) - \xi(0) = \xi(D)$ and the problem can be easily handled. In some sense, the problem of pricing a Parisian option with stochastic interest rates with a fixed time window to be spent below the level of the barrier relates to the price in the Black and Scholes framework of a Parisian option for which the activation window could change over time in a deterministic way. As far as we know, this has not been studied in the literature. The results in Lebart and Lelong (2009) are only valid if the activation time window is constant over time. However, it is possible to construct a lower bound and an upper bound of the price of the Parisian option. A similar idea also appears in Rogers and Shi (1995).

In general, ξ is a smooth deterministic function, so we can apply the Mean Value Theorem and have

$$\forall t \in (D, T), m := \inf_{u \in [D,T]} \{\xi'(u)\} D \leqslant \xi(t) - \xi(t - D) \leqslant M := \sup_{u \in [D,T]} \{\xi'(u)\} D.$$

This inequality is helpful to construct a lower bound and an upper bound for the price of a Parisian option under stochastic interest rates. This is because the price of the Parisian option is monotonic with the length of the time window. Intuitively, for "Up and In" Parisian call, the longer the time window, the "harder" for the option to be in the money, thus the lower the price of this option. Similar monotone relations hold between price and the length of the time window of the other types of Parisian options.

There are two particular cases. In the case when $\xi(t)$ is a concave function, then $\forall t \in [D,T], \xi(T)-\xi(T-D) < \xi(t)-\xi(t-D) < \xi(D)$. When $\xi(t)$ is a convex function, then $\forall t > 0, \xi(D) < \xi(t) - \xi(t-D) < \xi(T) - \xi(T-D)$. Here we prove the $\xi(t)$ being concave case and the other case can be similarly handled. If $\xi(t)$ is concave, then $\xi^{'}(t)$ is a decreasing function, so if we denote $g(t) = \xi(t) - \xi(t-D)$, then $g^{'}(t) = \xi^{'}(t) - \xi^{'}(t-D) < 0$, thus $g(t)$ is a decreasing function for $t \in [D,T]$. So $g(T) \leqslant g(t) \leqslant g(D)$ and we have the desired inequality.

Proposition 4.3.2 (Price of a Parisian Option with Stochastic Interest Rates)**.** *Let us denote by $PDO(f,D,L,T,r,\sigma)$, the price of a down and out Parisian option with payoff*

$$f(S_T)\mathbb{1}_{\hat{G}_{D,L}>T}$$

where D is the time window, L the level of the barrier, T the maturity of the option, $\hat{G}_{D,L}$ is the first time the process S has spent more than D continuously under the level L. The underlying process evolves as $dS_t = rS_t dt + \sigma S_t dW_t$ under Q and the interest rate is constant (Black and Scholes framework).

The price of a Parisian option with stochastic interest rates as given in (4.7) *with payoff*

$$f(A_T)\mathbb{1}_{G_{D,L}>T}$$

lies between

$$[PDO(f, \xi(T) - \xi(T-D), e^b, \xi(T), 0, 1), PDO(f, \xi(D), e^b, \xi(T), 0, 1)]$$

, where $b = \ln\left(\frac{LP(0,T)}{A_0}\right)$.

Proof From the discussion in Section 4.4, we know that in the Hull-White stochastic interest rate dynamic we assume as in equation (4.14),

$\xi(t)$ is always concave for $t \in [0, T]$. Thus from previous discussion we have $\forall t \in [D, T], \xi(T) - \xi(T-D) < \xi(t) - \xi(t-D) < \xi(D)$. Recall from equation (4.23) that $\{A_u = LP(u, T)\} = \{\widetilde{B}_{\xi(u)} = b\} = \{e^{\widetilde{B}_{\xi(u)}} = e^b\}$. So the hitting event of the asset A_t to the level $L(P(u, T))$ is equivalent to a geometric Brownian motion with drift 0 and volatility $\sigma = 1$ hitting the level e^b. Also note that for "Out" Parisian options, the price is monotonically increasing as the length of the time window increases. Then we have the desired bound. □

4.3.4 Comment on the quadratic variation $\xi(T)$

Actually for the purpose of numerical computation, the above lower and upper bounds are very close to their corresponding true value. Therefore we can approximate accurately the prices for Parisian options under stochastic interest rates.

The reason why the two bounds are tied is that in general ξ' is almost constant, in other words the absolute value of $\xi^{''}(t)$ is very small and often in the range of $[0, 2.0 * 10^{-4}]$. Just as the following graph shows, the second order derivative of $\xi(t)$ is close to 0. This means that the function $\xi(t)$ is approximately linear and we have a highly accurate approximation $x = \xi(t) - \xi(t-D) \approx \xi(t-t+D) = \xi(D)$. The precision is up to 3 decimal places and suitable for industry use. From the graph we can also see that $\xi(t)$ seems to be concave on the interval $[0, T]$ since its second order derivative is negative. This is justified in Section 4.4.

4.4 Property of function $\xi(t)$

In this section, we give further properties of the second order derivative of $\xi(t)$. For convenience of discussion, denote $g(t) = \xi^{''}(t)$, and $c = 1 + \rho\sigma\frac{a}{\nu}$, then $\rho\sigma\frac{a}{\nu} = c - 1$. Also note that since a, ν, ρ, σ are all positive, we have $c > 1$. Now we consider the first order derivative of $g(t)$ with respect to t

$$g^{'}(t) = 4\nu^2 e^{-2aT}\left(e^{at} - \frac{1}{2}ce^{aT}\right)e^{at} \tag{4.26}$$

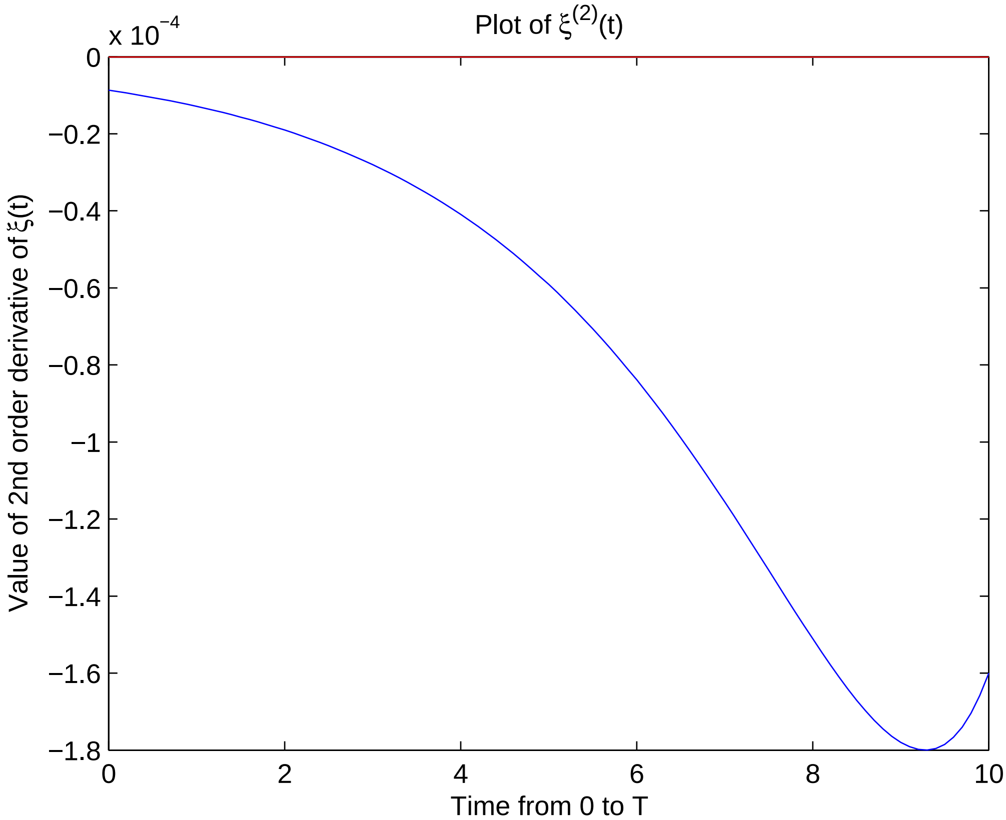

Figure 4.1: The plot of $\xi^{(2)}(t)$

Then we can find the critical point by letting $g^{'}(t) = 0$, solve it and we have

$$t^* = \frac{1}{a} \ln \left(\frac{1}{2} c \right) + T \tag{4.27}$$

since

$$\begin{aligned} g^{''}(t^*) &= 8a\nu^2 e^{-2aT} e^{2at^*} - 2\nu^2 cae^{-aT} e^{at^*} \\ &= a\nu^2 c^2 > 0 \end{aligned} \tag{4.28}$$

Thus by second order derivative test (4.28), we know that $g(t)$ attains its **minimum** at t^* if and only t^* falls inside the interval $[0, T]$. Note that from equation (4.27), "t^* falls inside the interval $[0, T]$" if and only if $c > 2$. Now we calculate the value of $g(t)$ at the two end points

$$\begin{aligned} g(0) = \xi^{''}(0) &= \frac{2\nu^2}{a} e^{-2aT} \left(1 - \frac{1}{2} c e^{aT} \right)^2 - \frac{\nu^2}{2a} c^2 \\ &= \frac{2\nu^2}{a} e^{-aT} \left(e^{-aT} - c \right) \end{aligned} \tag{4.29}$$

$$\begin{aligned} g(T) = \xi^{''}(T) &= \frac{2\nu^2}{a} e^{-2aT} \left(e^{aT} - \frac{1}{2} c e^{aT} \right)^2 - \frac{\nu^2}{2a} c^2 \\ &= \frac{2\nu^2}{a} (1 - c) \end{aligned} \tag{4.30}$$

Since there is only one critical point for $g(t)$ and it corresponds to a minimum, thus the maximum of $g(t)$ on interval $[0, T]$ will be at either 0 or T, thus we calculate the difference of the two values above

$$g(T) - g(0) = \frac{2\nu^2}{a} e^{-2aT} \left(e^{aT} - 1 \right) \left(1 - (c-1) e^{aT} \right) \tag{4.31}$$

Since $a > 0$, the sign of equation (4.31) will be determined by the sign of $1 - (c-1)e^{aT}$. Now we are ready to give a classification of the extreme points of function $g(t)$

Case 1 When $c > 2$, recall equation (4.27), we know that in this case $t^* > T$, thus the maximum or minimum will occur only at interval endpoints. Also note that $c > 2$ implies that $(c-1)e^{aT} > e^{aT} \geqslant 1$, thus $g(T) < g(0)$, so

$$\begin{aligned}\min_{t\in[0,T]} g(t) &= g(T) \\ &= \frac{2\nu^2}{a}(1-c) \\ &< \frac{2\nu^2}{a}(1-2) \\ &= -\frac{2\nu^2}{a}\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\max_{t\in[0,T]} g(t) &= g(0) \\ &= \frac{2\nu^2}{a}e^{-aT}\left(e^{-aT}-c\right) \\ &< -\frac{2\nu^2}{a}\end{aligned} \tag{4.33}$$

we can easily check that $\max_{t\in[0,T]} g(t) \geqslant \frac{2\nu^2}{a}(1-c) = \min_{t\in[0,T]} g(t)$.

Case 2 When $1 < c \leqslant 2$ and $e^{aT} \leqslant \frac{1}{c-1}$, we have the following

$$\begin{aligned}\min_{t\in[0,T]} g(t) &= g(t^*) \\ &= -\frac{\nu^2}{2a}c^2 \\ &\geqslant -\frac{2\nu^2}{a}\end{aligned} \tag{4.34}$$

and

$$\begin{aligned}\max_{t\in[0,T]} g(t) &= g(T) \\ &= \frac{2\nu^2}{a}(1-c) \\ &\geqslant -\frac{2\nu^2}{a}\end{aligned} \tag{4.35}$$

Also note that since $c > 1$, from the above equations (4.34) and (4.35), we know that $-\frac{2\nu^2}{a} \leqslant \min_{t\in[0,T]} g(t) \leqslant \max_{t\in[0,T]} g(t) < 0$.

Case 3 When $1 < c \leqslant 2$ and $e^{aT} > \frac{1}{c-1}$, we have the following

$$\begin{aligned}\min_{t\in[0,T]} g(t) &= g(t^*) \\ &= -\frac{\nu^2}{2a}c^2 \\ &\geqslant -\frac{2\nu^2}{a} \end{aligned} \tag{4.36}$$

$$\begin{aligned}\max_{t\in[0,T]} g(t) &= g(0) \\ &= \frac{2\nu^2}{a}e^{-aT}\left(e^{-aT}-c\right) \\ &\geqslant \frac{2\nu^2}{a}(c-1)(c-1-c) \\ &= -\frac{2\nu^2}{a}(c-1) \\ &\geqslant -\frac{2\nu^2}{a} \end{aligned} \tag{4.37}$$

Also note that since $c > 1$, from the above equations (4.36) and (4.37), we know that $-\frac{2\nu^2}{a} \leqslant \min_{t\in[0,T]} g(t) \leqslant \max_{t\in[0,T]} g(t) < 0$

Remark Above all, in all three cases we have that the extremes of the function $g(t)$ in the interval $[0,T]$ are **negative**. This means that $g(t) < 0$ **for all** $t \in [0,T]$. So the function $\xi(t)$ is **always concave** in the interval $[0,T]$.

1.When $c \leqslant 2$, $-\frac{2\nu^2}{a} \leqslant \min_{t\in[0,T]} g(t) \leqslant \max_{t\in[0,T]} g(t) < 0$. Thus we need to **minimize** $\frac{2\nu^2}{a}$ as much as possible in order to ensure that the second order derivative of $\xi(t)$ is small.

2.When $c > 2$, we have to control $\frac{2\nu^2}{a}(c-1)$ and $\frac{2\nu^2}{a}e^{-aT}\left(c-e^{-aT}\right)$ given that we have $\frac{\nu}{a} < \rho\sigma$ or $c > 2$.□

4.5 Conclusion of Chapter 4

In this chapter, we have managed to price Parisian options with stochastic interest rates in the case of a stochastic discounted barrier. The main idea is the use of the **Dubins Schwarz Theorem** and of the **Girsanov**

Theorem. The change of measure allows us to reduce the problem to the pricing of Parisian options in Black and Scholes framework. Also we have to mention that (4.16) is vital to the development. It says that the function $\xi(t)$ is monotonically increasing. (4.16) holds under the assumption of an exponential volatility structure (Hull-White framework). Since we do not have a one to one correspondence for intervals under the original clock and the new clock, we can only derive bounds for the true Parisian option price under stochastic interest rates. However, we numerically verify that this bound is quite close by plotting the function $\xi(t), t \in [0, T]$ for a certain parameter set. The same idea can also be applied to pricing options involving "time intervals" under stochastic interest rates, e.g. cumulative Parisian options, Step options or any other options related to the occupation time of the underlying asset.

Our method works for any volatility structure as long as equation (4.16) is satisfied. Then we will get a one to one mapping between the original time and the transformed one. Then we will have equivalence between events for these two times in ((4.23), which plays a vital role. We are only dealing with the case of discounted barrier (refer to equation (4.23)), one possible extension is to consider cases when the barrier is constant. Difficulty arises and from what we observe, it seems that this will *inevitably* involve more than two time-changed Brownian motions, one for the asset process and the other for the bond process. Barrier option pricing under Stochastic interest rates with a constant barrier is thus left for future research.

We left for future research to move to a more realistic default model such as described in (4.4). We also need to study the sensitivity of the bounds to the parameters. Intuitively there are two parameters that can explain the accuracy. First the bounds will be a good approximation of the price of the Parisian option under stochastic rates if the bounds on the time window are tight and if the price is not very sensitive to changes in the length of the time window D.

Chapter 5

Timer option pricing

In this chapter, we discuss a newly introduced exotic derivative called the "Timer Option". Instead of being exercised at a fixed maturity date as a vanilla option, it has a random date of exercise linked to the accumulated variance of the underlying stock. In the case of the Hull and White and of the Heston stochastic volatility models, we propose a fast and accurate method to price these securities using the stochastic time change idea. The approach is based on Antonelli and Scarlatti (2009)'s Taylor expansion with respect to the correlation between the underlying stock and its variance. Then we discuss the pricing and the practical interests of the timer-style options available in the marketplace, namely the capped timer option, the FX timer option, the time swap and the timer out-performance option. Finally we propose several new designs of timer-style options.

Introduction

Recently (in April 2007), Société Générale Corporate and Investment Banking (SG CIB) started to sell a new type of option that allows buyers to "specify the level of volatility used to price the instrument". The standard version of this new product is called a "timer call". A timer call is similar to a call option with a random maturity date determined by the time needed for the accumulated variance of the underlying stock to reach a prespecified level. So far, these products have not been given much attention by practitioners or by academics. Carr and Lee (2009) mention that they are also known as "mileage" options. Bick (1995) is the first author who worked on timer options although they did not exist at that time[1]. In the case when $r = 0\%$ there exist robust replications for timer options as a special case of the general quadratic variation derivatives studied by Carr and Lee (2010)(2009). Recently Li (2009a)(2009b) explains how to price and hedge these new challenging securities in the Heston stochastic volatility model. In this paper, we propose simple and accurate methods to price timer call options in more general stochastic volatility models. We discuss

[1]Bick (1995) writes "It should be emphasized that it is *not* the purpose of this paper to price options that do not exist in practice."

their properties and study existing “timer-style” financial derivatives as well as propose and price more exotic timer style options.

Practical implications for timer-style options seem very important. Sawyer (2007) explains that “*this product is designed to give investors more flexibility and ensure they do not overpay for an option. The price of a vanilla call option is determined by the level of implied volatility quoted in the market, as well as maturity and strike price. But the level of implied volatility is often higher than realized volatility, reflecting the uncertainty of future market direction. In simple terms, buyers of vanilla calls often overpay for their options. In fact, having analyzed all stocks in the Euro Stoxx 50 index since 2000, SG CIB calculates that 80% of three-month calls that have matured in-the-money were overpriced.*” Timer options are linked to the realized volatility of some underlying index, stock or exchange rate. Due to their apparent complexity, “timer options” were first sold to sophisticated investors such as hedge funds but are more and more widely sold.

This work has several contributions. We provide a comprehensive study of the timer options that exist in the market, discuss practical implications that timer options may have in the future and propose new designs of exotic timer-style derivatives. We also develop fast and accurate methods to price timer options in some stochastic volatility models. In the case of the Heston model, it extends the work by Li (2009a)(2009b). We also investigate the Hull and White stochastic volatility model in details. Our approximation formula is based on Antonelli and Scarlatti (2009) and is valid in very general stochastic volatility models. Moreover there are several cases when the pricing formula for timer options can be simplified significantly, such as when the correlation between the underlying stock's returns and its volatility process is equal to zero or when the risk-free interest rate is equal to zero. Finally, we discuss the sensitivity of the timer options to the correlation coefficient between the asset and its volatility.

The first section is dedicated to the standard timer call option and its pricing. The second section illustrates the theoretical study by some numerical examples and discusses the dependence on the correlation coefficient ρ. The third section gives some practical implications of timer options and presents more recent timer-style contracts that SG CIB and Lehman Brothers started to sell including the capped timer call option, the FX timer call, the "timer out-performance" option and the "time swap". In the fourth section we propose new "timer-style" contracts that could be of interest to investors. As far as we know, the contracts proposed in the last section are not yet commercialized.

5.1 Timer Call

In this section, we explain in details what a standard timer call is, how it actually works, and we propose a new approach for pricing this product in the Heston and the Hull and White stochastic volatility models.

5.1.1 Standard Timer Call

A standard timer call option can simply be viewed as a call option with random maturity which depends on the time needed for a pre-specified variance budget to be fully consumed. With the timer call, the buyer can specify an investment horizon and a target volatility. A variance budget is then calculated as the target volatility squared, multiplied by the target maturity. Once the variance budget is consumed, the option expires. Let us now summarize practical details of timer options obtained from a presentation of the Société Générale (2007).

The investor chooses a target volatility Σ (also called implied volatility target) and a maturity T to establish a fixed variance budget

$$VB^{Target} = \Sigma^2 \frac{N_T}{252} \tag{5.1}$$

where N_T is the number of trading days before the maturity date T. The realized volatility for the observation period D is calculated as

$$\Sigma_D^{realized} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2} \tag{5.2}$$

where $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) = \ln(S_i) - \ln(S_{i-1})$, and where S_i is the observed underlying stock at time t_i, and $0 < t_1 < ... < t_n = D$ (the discretization step is daily in practice). To scale for a one-year period, we divide by $\sqrt{D}$, and the *annualized* realized volatility becomes

$$\sigma_{realized} = \frac{\Sigma_D^{realized}}{\sqrt{D}}.$$

As the stock moves daily, the variance budget is "expended" according to the *realized variance consumption* formula

$$VB^{realized} = \sigma^2_{realized} \frac{d}{252} \tag{5.3}$$

where d is the number of days since the inception date. When the realized variance consumption $VB^{realized}$ is larger than the variance budget target VB^{Target}, the option is automatically exercised. If the realized volatility exactly matches the investor's target volatility, the expiry of the option will equal the target investment horizon. If the realized volatility is higher or lower, the option will expire respectively at an earlier or later date.

A timer call option was first traded at the end of April 2007 with a hedge fund. *" At the time, the implied volatility on the plain vanilla call was slightly above 15%, but the client set a target volatility level of 12%, a little higher than the prevailing realized volatility level of around 10%. By rolling into a timer call, the hedge fund reduced its premium by 20%.Since the inception of the trade, the realized volatility has been around 9.5%. If it remains at this level, the maturity of the timer call will be 60% longer than the original vanilla call."* (see Sawyer (2007)).

Since then, SG CIB has started to sell two new timer-style options called respectively the "timer out-performance option" and the "time swap" that we will describe and discuss in section 5.3.2. After investors become more familiar with the pros and cons of *timer* options, one can expect that many new and more exotic timer-style contracts will be issued. We propose some new "timer-style" options later in section 5.4.

Timer options are very innovative products. Traditionally financial derivatives or hedging strategies consist of hedging a payoff at a fixed maturity. Using timer options means adopting a very different viewpoint.

Now the maturity is random, and the investment strategy is driven by the quadratic-variation. Some thoughts about quadratic-variation-based strategies appeared already more than 10 years ago among academics (for instance Bick (1995), Geman and Yor (1993), Rendleman and O'Brien (1990)). Quadratic-variation-based strategies can be cheaper hedging strategies, and are very promising. We now present the setting in which we derive the prices of timer options.

5.1.2 Model

Let us consider a standard timer call option with strike K and written on an underlying asset S. Its maturity date is linked to the accumulated variance of this underlying stock. We assume that the stock evolves according to a stochastic volatility model, and that the interest rate r is constant. Under a risk neutral measure Q, which we will use for pricing, one has

$$\begin{cases} dS_t &= rS_t dt + \sqrt{V_t} S_t \left(\sqrt{1-\rho^2} dW_t^1 + \rho dW_t^2 \right) \\ dV_t &= \alpha(V_t) dt + \beta(V_t) dW_t^2 \end{cases} \tag{5.4}$$

where $\alpha(.)$ and $\beta(.)$ are deterministic functions, W^1 and W^2 are independent Brownian motions and ρ reflects the correlation between the stock returns and the changes in the stock's volatility process. In the Heston model (Heston (1993)), the variance process evolves as

$$dV_t = \kappa \left(\theta - V_t \right) dt + \sigma_v \sqrt{V_t} dW_t^2 \tag{5.5}$$

where κ, θ and σ_v are the parameters of the volatility process, they are positive constants. We assume that the Feller condition $2\kappa\theta - \sigma_v^2 > 0$ is satisfied (See Revuz and Yor (2005)). In the Hull and White model (Hull and White (1987)),

$$dV_t = \mu_v V_t dt + \xi_v V_t dW_t^2 \tag{5.6}$$

where μ_v and ξ_v are two positive constants.

When buying a timer option, the investor would specify a target volatility and a target investment horizon to calculate the "variance budget". Let us denote by $\mathbb{V}$ this constant "variance budget" that is chosen by the investor (corresponding to VB^{Target} given by (5.1)). The stock price evolves in a continuous time framework (see (5.4)), we define the realized variance in continuous time at time u as

$$\int_0^u V_s ds,$$

corresponding to the realized variance consumption (given by $VB^{realized}$ in (5.3)). Denote by τ the random maturity time of the option. It is defined as the first hitting time of the accumulated variance to the variance budget $\mathbb{V}$

$$\tau = \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}\right\}. \tag{5.7}$$

In the above framework, the price of the timer call option is given by

$$C_0 = E^Q\left[e^{-r\tau}\max\left(S_\tau - K, 0\right)\right] \tag{5.8}$$

where Q denotes the risk-neutral probability under which S follows (5.4).

5.1.3 Theory of Pricing Timer Options

To simplify the expression of the price (5.8), we use the "time change technique" (see for instance Geman (2008)). To do so, we introduce N, the following martingale process

$$N_T = \int_0^T \sqrt{V_t}\left(\sqrt{1-\rho^2}dW_t^1 + \rho dW_t^2\right).$$

Note that N_T can be decomposed into a linear combination of two martingale processes ($\int_0^T \sqrt{V_t}dW_t^1$ and $\int_0^T \sqrt{V_t}dW_t^2$) with the same quadratic-variation $\xi(T)$ at time T

$$\xi(T) = \int_0^T V_t dt. \tag{5.9}$$

We now apply Theorem 4.6 on page 174 of Karatzas and Shreve (1991), which is also often referred as Dubins-Schwarz Theorem or time-change technique for martingales. Since $\lim_{T\to\infty}\xi(T)=\infty$ (in the Heston model and the Hull and White model), we can use this theorem and state that under the measure Q, there exist two standard Brownian motions B^1 and B^2 such that

$$N_T = B_{\xi(T)} = \sqrt{1-\rho^2}B^1_{\xi(T)} + \rho B^2_{\xi(T)}. \tag{5.10}$$

where $B^i_{\xi(T)} = \int_0^T \sqrt{V_t}dW^i_t$ for $i=1,2$. Since W^1 and W^2 are independent, B^1 and B^2 are also standard independent Brownian motions. Therefore, the process B is also a standard Brownian motion. The underlying asset S_t at time t can be written as

$$S_t = S_0 e^{rt} e^{B_{\xi(t)}-\frac{1}{2}\xi(t)}. \tag{5.11}$$

By the definition of τ, we have

$$\xi(\tau) = \mathbb{V}. \tag{5.12}$$

Using this property in equation (5.11), the stock price at the exercise time of the timer call option can be written as

$$S_\tau = S_0 e^{r\tau} e^{B_{\xi(\tau)}-\frac{1}{2}\xi(\tau)} = S_0 e^{r\tau} e^{B_{\mathbb{V}}-\frac{1}{2}\mathbb{V}} \tag{5.13}$$

and the price of the timer call at time 0 is now equal to

$$C_0 = E^Q\left[\max\left(S_0 e^{B_{\mathbb{V}}-\frac{1}{2}\mathbb{V}} - Ke^{-r\tau}, 0\right)\right]. \tag{5.14}$$

Note that $B_{\mathbb{V}}$ and τ can be correlated. Indeed, when $\rho \neq 0$, it can easily be seen from the formula (5.10) replacing T by τ that $B_{\mathbb{V}}$ and B^2 are dependent. Since τ is determined by the trajectory of the variance process, and therefore of B^2, then $B_{\mathbb{V}}$ and τ may be dependent.

Remark 5.1.1. *There is a very interesting special case when the interest rate is equal to $r=0\%$. In this case, the formula* (5.14) *can be simplified and the price of a timer call option is given by*

$$C_{0|r=0\%} = E^Q\left[\max\left(S_0 e^{B_{\mathbb{V}}-\frac{1}{2}\mathbb{V}} - K, 0\right)\right].$$

In this case, the price of a timer call option has a closed-form expression equal to the Black and Scholes formula (with initial underlying stock price

S_0, interest rate $r = 0\%$, and such that σ the constant volatility in Black and Scholes model and T the maturity of the call option are such that $\sigma^2 T = \mathbb{V}$). It is equal to

$$C_{0|r=0\%} = S_0 \mathcal{N}\left(\widehat{d_1}\right) - K\mathcal{N}\left(\widehat{d_2}\right) \tag{5.15}$$

where $\widehat{d_1} = \frac{\ln\left(\frac{S_0}{K}\right)+\frac{1}{2}\mathbb{V}}{\sqrt{\mathbb{V}}}$ and $\widehat{d_2} = d_1 - \sqrt{\mathbb{V}}$.

This last remark is quite intuitive. The difference between the timer option and a standard option comes from the maturity date of the contract which is random. When the interest rate is equal to 0%, then the exact dates when each cash-flow occurs do not really matter. It is therefore intuitive that the price of the timer option does not depend on τ anymore. In this case, the result does not depend on the assumptions on the stochastic volatility model, and there exist robust replications for a wide range of volatility derivatives (Carr and Lee (2010)), this applies in the case of timer options (Bick (1995)).

Note also that Formula (5.15) must be the limit case of the formula 4.1 in Theorem 1 of Li (2009b) or formula 3.1 in Theorem 1 of Li (2009a) when $r = 0\%$.

Remark 5.1.2. *In the case when the volatility is deterministic or constant, τ is deterministic and the formula of Black and Scholes holds with $T = \tau$, with initial underlying stock price S_0, interest rate r, the underlying's volatility σ such that $\sigma^2 T = \mathbb{V}$.*

Remark 5.1.3. *Prices for timer put options can be obtained from the timer call prices thanks to a put-call parity formula for timer options $C_0 - P_0 = S_0 - KE^Q(e^{-r\tau})$ (See for instance Li (2009a)(2009b)).*

Remark 5.1.4. *Formula* (5.14) *holds under general assumptions for the volatility process. In particular we only use the fact that $\xi(T)$ defined by* (5.9) *exists and goes to $+\infty$ when $T \to +\infty$.*

5.1.4 Approximating Timer Options Prices

We now need to investigate how to evaluate formula (5.14), and how to handle the dependence between τ and $B_{\mathbb{V}}$ in this formula. At first, it seems that it is a two-dimension problem since there are two independent Brownian motions that contribute to the payoff of a timer option (W_1 and W_2).

First, we show how to price timer options when the volatility is modeled by the Heston stochastic volatility model. In this case, the price of a timer option can be expressed as a function of (τ, V_τ) and therefore simulated from the distribution of (τ, V_τ). It only depends on the trajectories of B^2 and therefore it becomes a one-dimension problem. This is a faster and more accurate technique to deal with the pricing of the timer option in the Heston stochastic volatility model than using Li (2009a)'s formula (3.1). This formula indeed involves a semi-closed-form of the joint distribution of τ and V_τ involving a multidimensional integral of a complex function with infinite bounds. At the same time, we show that the effect of the correlation coefficient on the price of timer options is weak.

Second, without any assumption on the stochastic volatility model, in the particular case when the volatility process is not correlated to the underlying stock returns, the price of a timer option depends only on the dynamics of the variance process and can be reduced to a one-dimension problem but this is not true in general when $\rho \neq 0$.

When there is some correlation, the problem can be solved in the Heston model but not easily in other stochastic volatility models. In more general cases, we give a Taylor expansion with respect to the correlation coefficient using a result by Antonelli and Scarlatti (2009). For the purpose of illustration, accurate approximations are derived in the Heston model as well as in the Hull and White model.

Heston Model

To understand the impact of the correlation coefficient between the variance process and the underlying, we express (5.13) using the decomposition of B as a linear combination of the two independent Brownian motions B^1 and B^2, Formula (5.13) shall become

$$S_\tau = S_0 e^{r\tau + \rho B^2_{\mathbb{V}} - \frac{1}{2}\mathbb{V} + \sqrt{1-\rho^2} B^1_{\mathbb{V}}}. \tag{5.16}$$

The price of a timer call option can then be written as

$$C_0 = E^Q \left[E^Q \left[\max \left(S_0 e^{\rho B^2_{\mathbb{V}} - \frac{1}{2}\mathbb{V}} e^{\sqrt{1-\rho^2} B^1_{\mathbb{V}}} - K e^{-r\tau}, 0 \right) \middle| \tau, V_\tau \right] \right]. \tag{5.17}$$

However $B^2_{\mathbb{V}} = \int_0^\tau \sqrt{V_t} dW^2_t$. In the case of the Heston stochastic volatility model, it is possible to write

$$B^2_{\mathbb{V}} = B^2_{\mathbb{V}}(\tau, V_\tau) = \frac{V_\tau - V_0 - \kappa\theta\tau + \kappa\mathbb{V}}{\sigma_v}, \tag{5.18}$$

(see appendix 5.5). Since B^1 is independent of B^2 and therefore of (τ, V_τ), the conditional distribution of $B^1_{\mathbb{V}}$ given (τ, V_τ) is equal to the unconditional distribution, that is, its conditional distribution is still $\mathcal{N}(0, \mathbb{V})$. Therefore the price of a timer call option is given by

$$C_0 = S_0 E^Q \left[e^{\rho B^2_{\mathbb{V}} - \frac{1}{2}\rho^2 \mathbb{V}} \mathcal{N}\left(d_1(\tau, V_\tau)\right) \right] - K E^Q \left[e^{-r\tau} \mathcal{N}\left(d_2(\tau, V_\tau)\right) \right], \tag{5.19}$$

where

$$d_1(\tau, V_\tau) = \frac{\ln\left(\frac{S_0}{K}\right) + r\tau + \rho B^2_{\mathbb{V}} + \left(\frac{1}{2} - \rho^2\right)\mathbb{V}}{\sqrt{(1-\rho^2)\mathbb{V}}}, \quad d_2(\tau, V_\tau) = d_1(\tau, V_\tau) - \sqrt{(1-\rho^2)\mathbb{V}},$$

and $B^2_{\mathbb{V}}$ is given by its formula (5.18).

This expression (5.19) can be evaluated by Monte Carlo simulations. To do so, we need to simulate n times (τ, V_τ) under Q to obtain iid samples $(\tau_i, V_{\tau_i})_{i=1..n}$. Then an approximation of the price of the timer call option can be obtained by crude Monte Carlo techniques by

$$C^{mc} = \frac{S_0 e^{-\frac{1}{2}\rho^2\mathbb{V}}}{n} \sum_{i=1}^{n} e^{\rho B^2_{\mathbb{V}}(\tau_i, V_{\tau_i})} \mathcal{N}\left(d_1(\tau_i, V_{\tau_i})\right) - \frac{K}{n} \sum_{i=1}^{n} e^{-r\tau_i} \mathcal{N}\left(d_2(\tau_i, V_{\tau_i})\right) \tag{5.20}$$

where $B^2_{\mathbb{V}}(\tau_i, V_{\tau_i})$ is defined by (5.18). We obtain an iid sample of (τ, V_τ) using the joint law of (τ, V_τ)

$$(\tau, V_\tau) \overset{law}{\sim} \left(\int_0^{\mathbb{V}} \frac{ds}{\sigma_v X_s},\ \sigma_v X_{\mathbb{V}} \right), \tag{5.21}$$

where X_t is a standard Bessel process

$$dX_t = \left(\frac{\kappa\theta}{\sigma_v^2 X_t} - \frac{\kappa}{\sigma_v} \right) dt + dB_t, \;\; X_0 = \frac{V_0}{\sigma_v}, \tag{5.22}$$

and where B a standard Brownian motion (see for example Proposition 5 of Li (2009a)). More details about this process can also be found in Linetsky (2004).

Special Case $\rho = 0$

When $\rho = 0$, that is there is no correlation between the variance process and the stock returns, then τ and $B_{\mathbb{V}}$ are independent. This case is standard in credit risk modelling, see for instance section 5 of Packham, Schlögl and Schmidt (2009).

First let us simplify the expression (5.14) of the price of a timer call by conditioning with respect to the random variable τ.

$$C_0 \;\; = \;\; E^Q \left[E^Q \left[\max \left(S_0 e^{B_{\mathbb{V}} - \frac{1}{2}\mathbb{V}} - K e^{-r\tau}, 0 \right) \Big| \, \tau \right] \right]. \tag{5.23}$$

After simplifying the above expression (5.23), the price of a timer call option (when $\rho = 0$) is given by

$$C_{0|\rho=0} \;\; = \;\; S_0 E^Q \left[\mathcal{N}\left(d_1(\tau)\right) \right] - K E^Q \left[e^{-r\tau} \mathcal{N}\left(d_2(\tau)\right) \right] \tag{5.24}$$

where

$$d_1(\tau) = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\mathbb{V} + r\tau}{\sqrt{\mathbb{V}}}, \quad d_2(\tau) = d_1(\tau) - \sqrt{\mathbb{V}}.$$

This expression (5.24) can be evaluated by Monte Carlo simulations. To do so, we need to simulate n times τ under Q to obtain an iid sample $(\tau_i)_{i=1..n}$. Then an approximation of the price of the timer call option can be obtained by crude Monte Carlo given n simulations of τ

$$C^{mc} = \frac{S_0}{n} \sum_{i=1}^{n} \mathcal{N}\left(d_1(\tau_i)\right) - \frac{K}{n} \sum_{i=1}^{n} e^{-r\tau_i} \mathcal{N}\left(d_2(\tau_i)\right). \tag{5.25}$$

Depending on the assumptions about the stochastic volatility model, an iid sample of τ can be obtained by different techniques. First it could be directly simulated from the discretized variance process using for example an Euler scheme (for the second line of the equation (5.4)) but this could lead to very lengthy simulations. The random variable τ can also be simulated from the distribution of τ when it is known which depends on the assumption on the dynamics of the volatility process.

Heston Model

In the case of the Heston stochastic volatility model, one can make use of (5.21) to simulate τ and proceed as before.

Hull and White Model

In the Hull and White model, the variance process evolves as in (5.6). To simulate τ, there are two possible approaches. First we can use a similar technique as in the Heston case to establish a similar expression as in (5.21). The joint law of (τ, V_τ) is

$$(\tau, V_\tau) \overset{law}{\sim} \left(\frac{4}{\xi_v^2} \int_0^{\mathbb{V}} \frac{ds}{X_s^2}, \frac{\xi_v^2}{4} X_{\mathbb{V}}^2 \right), \tag{5.26}$$

where X_t is governed by

$$dX_t = \left(\frac{2\mu_v}{\xi_v^2} - \frac{1}{2} \right) \frac{1}{X_t} dt + dB_t, \quad X_0 = \frac{2}{\xi_v} \sqrt{V_0}. \tag{5.27}$$

where B a standard Brownian motion. We can immediately see that X_t is a standard Bessel process with index $v = \frac{2\mu_v}{\xi_v^2} - 1$. See section 5.6 for details to obtain (5.26).

Second, we can use the semi-closed-form expression for the distribution of τ. There exists a closed-form expression of the Laplace transform of the density of the stopping time τ (see section 4 of Geman and Yor (1993)). Let g denote the density of τ. Its Laplace transform is given by

$$\int_0^\infty g(x) e^{-\lambda x} dx = \frac{1}{\Gamma(k)} \int_0^1 \exp\left\{ \frac{-uV_0}{2\mathbb{V}\eta^2} \right\} \left(\frac{uV_0}{2\mathbb{V}\eta^2} \right)^k (1-u)^{\mu_2 - k} du. \tag{5.28}$$

where

$$v = \frac{2\mu}{\xi^2} - 1; \quad \mu_2 = \left(\frac{2\lambda}{\eta^2} + v^2 \right)^{\frac{1}{2}}; \quad k = \frac{\mu_2 - v}{2}, \quad \eta = \frac{\xi}{2}.$$

One can invert (5.28) and obtain g using for instance Abate and Whitt (1995) Laplace inversion technique (see also Geman and Eydeland (1995)). Then one can simulate τ with the density g or perform a numerical integration since the price of a timer call option given by (5.24) can now be expressed as an integral

$$C_0 = S_0 \int_0^{\infty} \mathcal{N}(d_1(x)) g(x) dx - K \int_0^{\infty} \mathcal{N}(d_2(x)) e^{-rx} g(x) dx \quad (5.29)$$

where $g(x)$ is the density function of τ. The timer option is then priced by a semi-closed-form formula. Numerical examples can be found in section 5.2.

Remark 5.1.5. *Formula* (5.24) *holds only when* $B_{\mathbb{V}}$ *conditioned with respect to* τ *is normally distributed* $\mathcal{N}(0, \mathbb{V})$.

Using simulations and some realistic examples in the case of the Heston model, we verified numerically that the dependence between the correlation coefficient ρ and the timer option prices is very weak (see section 5.2.2). The fact that this dependence is very weak has several important consequences. First the price (5.24) is a rough approximation of the real price. Second hedging timer call option may be possible using only one instrument. Finally it shows that timer call options will not be good instruments to hedge the correlation between the stock and its variance process.

In the case when $\rho \neq 0$, there is a simple way of significantly improving the rough approximation (5.24) that we explain in the next paragraph.

General case $\rho \neq 0$

We now explain how to obtain an accurate approximation of the price of the timer option when the correlation is not equal to 0. To do so we make use of recent papers by Alòs (2006) and Antonelli and Scarlatti (2009). The latter paper enables us to approximate the price of a timer option under very general stochastic volatility models including Heston model, Hull-White model, Shoebel-Zhu model, and Stein-Stein model.

The method consists of studying a Taylor expansion of the price of a call option with respect to the correlation parameter ρ. Using the main development of Antonelli and Scarlatti (2009), the price of a timer option, C_0 (given by (5.14)) can be expressed using the case of zero correlation, precisely $C_{0|\rho=0}$ given by (5.24), as follows

$$C_0 \approx C_{0|\rho=0} + E_Q\left[\widehat{g}_1(\tau)\right]\rho, \tag{5.30}$$

where $\widehat{g}_1$ depends on the assumed stochastic volatility model. To calculate an approximation of the price of the timer call option, we then only need to simulate τ, and use the closed-form formula for $\widehat{g}_1$ given by Antonelli and Scarlatti (2009). We now give the expression of $\widehat{g}_1$ in the Heston model (variance process evolves as (5.5)) and in the Hull and White model (variance process evolves as (5.6)).

Heston Model Applying the results of Antonelli and Scarlatti (2009) in our setting,

$$\widehat{g}_1^H(\tau) = -\frac{\sigma_v K e^{-r\tau}\widehat{d}_2\mathcal{N}'(\widehat{d}_2)}{2\kappa\mathbb{V}}\left(\frac{V_0-\theta}{2\kappa}\left(1-e^{-\kappa\tau}\right)+\tau\left(\theta-(V_0-\theta)e^{-\kappa\tau}\right)\right) \tag{5.31}$$

where

$$\widehat{d}_2 = \frac{\ln(\frac{S_0}{K}) + r\tau - \frac{1-\rho^2}{2}\mathbb{V}}{\sqrt{(1-\rho^2)\mathbb{V}}}, \tag{5.32}$$

and $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

In the numerical example (in section 5.2), we compare the approximation (5.30) with the price obtained by (5.19) using Monte Carlo simulations of (τ, V_τ). The approximation (5.30) contains a bias (because it contains only the first-order Taylor expansion with respect to the correlation coefficient ρ), the formula (5.19) converges to the true price when the number of simulations goes to infinity and the discretization step tends to 0. The numerical example confirms the accuracy of this approximation. Similar to Antonelli and Scarlatti (2009), we found that at least the first digit is correct.

Hull-White model

From the result on page 298 of Antonelli and Scarlatti (2009), we can also adapt their result and obtain it for our setting. In this case,

$$\widehat{g}_1^{HW}(\tau) = -\frac{\xi_v V_0 K e^{-r\tau}\widehat{d}_2\mathcal{N}'(\widehat{d}_2)}{e^{(2\mu_v+\xi_v^2)\tau}-1}\left[\frac{e^{(3\mu_v+3\xi_v^2)\tau}(2\mu_v+\xi_v)}{(\mu_v+2\xi_v^2)(3\mu_v+3\xi_v^2)} - \frac{e^{(2\mu_v+\xi_v^2)\tau}}{\mu_v+2\xi_v^2} + \frac{1}{3\mu_v+3\xi_v^2}\right] \tag{5.33}$$

where $\widehat{d}_2$ is also given by (5.32).

Since a semi-closed-form expression is known for the distribution of τ (given by (5.28)), the formula (5.30) is also a semi-closed-form expression for an approximation of the price of the timer option.

Using Antonelli and Scarlatti (2009) approach, we are able to obtain the price of a timer call option by using only the distribution of τ. The problem becomes therefore a one-dimension problem since it is solely determined by the dynamics of the variance process (it only depends on the dynamics of the Brownian motion B^2).

5.1.5 Greeks of the timer call option

This section solves for the greeks in the Heston stochastic volatility model. The hedging parameters follow easily from equations (5.16) and (5.17). Given (τ, V_τ), we can write the asset price as

$$S_\tau = \left(S_0 e^{\rho B^2_{\mathbb{V}}(\tau,V_\tau)-\frac{\rho^2\mathbb{V}}{2}}\right) e^{r\tau-\frac{1}{2}\widehat{\sigma}_\rho^2\tau+\widehat{\sigma}_\rho\sqrt{\tau}Z} = \widehat{S}_0 e^{r\tau-\frac{1}{2}\widehat{\sigma}_\rho^2\tau+\widehat{\sigma}_\rho\sqrt{\tau}Z}$$

where Z is a standard normal distribution $\mathcal{N}(0,1)$ and

$$\widehat{S}_0 = S_0\xi_\tau,\ \xi_\tau = e^{\rho B^2_{\mathbb{V}}(\tau,V_\tau)-\frac{\rho^2\mathbb{V}}{2}},\ \widehat{\sigma}_\rho = \sqrt{\frac{(1-\rho^2)\mathbb{V}}{\tau}}. \tag{5.34}$$

To calculate the greeks of a timer call option, we first note that its price can be expressed using the Black and Scholes formula for the standard European call option

$$C_0 = E\left[E\left[C_{BS}\left(\widehat{S}_0, K, r, \tau, \widehat{\sigma}_\rho\right) \mid \tau, V_\tau\right]\right] \tag{5.35}$$

where $C_{BS}(\widehat{S}_0, K, r, \tau, \sigma)$ denotes the price of a call option in the Black and Scholes setting where the initial underlying's price is equal to $\widehat{S}_0$, the strike is K, the interest rate is constant equal to r, the maturity is τ and the underlying's volatility σ. Then under some regularity conditions, we obtain the following expression for the delta of a timer call option

$$\Delta_c = \frac{\partial}{\partial S_0} C_0 = E\left[E\left[\Delta_{BS}(\widehat{S}_0, K, r, \tau, \widehat{\sigma}_\rho)\xi_\tau \mid \tau, V_\tau\right]\right] \tag{5.36}$$

where Δ_{BS} denotes the delta of a call option in the Black and Scholes framework. Similarly one can obtain an expression for the gamma and the vega of the timer call option

$$\Gamma_c = \frac{\partial^2 (C_0)}{\partial S_0^2} = E\left[E\left[\Gamma_{BS}(\widehat{S}_0, K, r, \tau, \widehat{\sigma}_\rho)\xi_\tau^2 \mid \tau, V_\tau\right]\right] \tag{5.37}$$

where Γ_{BS} denotes the gamma of a call option in the Black and Scholes framework. The sensitivity to the correlation coefficient is given by

$$\frac{\partial C_0}{\partial \rho} = E\left[E\left[\Delta(\widehat{S}_0, K, r, \tau, \widehat{\sigma}_\rho)\frac{\partial}{\partial \rho}\xi_\tau - \frac{\rho\sqrt{\mathbb{V}}}{\sqrt{(1-\rho^2)\tau}}\frac{\partial C_{BS}}{\partial \rho}(\widehat{S}_0, K, r, \tau, \widehat{\sigma}_\rho) \mid \tau, V_\tau\right]\right]$$

Finally, the sensitivity to the interest rate, Rho, can be obtained from this expression

$$\frac{\partial C_0}{\partial r} = E\left[E\left[Rho_{BS}(\widehat{S}_0, K, r, \tau, \widehat{\sigma}_\rho) \mid \tau, V_\tau\right]\right] \tag{5.38}$$

where Rho_{BS} denotes the rho of a call option in the Black and Scholes setting.

5.1.6 Pricing at time t

In the above paragraphs, we studied how to price timer call options at the inception date. It is actually easy to extend the pricing formula to a later date t. The important variable is the "consumed variance" at the valuation date t which can be calculated as

$$CV_t = \int_0^t V_s ds.$$

There are two cases, if CV_t exceeds the variance budget target $\mathbb{V}$, then the option has already expired and therefore has no value at time t. Otherwise, one has $t < \tau$, and the price of the timer call option at time t can be calculated as

$$C_0 = E^Q \left[e^{-r(\widetilde{\tau}-t)} \max\left(S_{\widetilde{\tau}} - K, 0\right) \right]$$

where $\tilde{\tau}$ is now defined as follows

$$\widetilde{\tau} = \inf \left\{ u > t, \int_t^u V_s ds = \mathbb{V} - CV_t \right\}. \tag{5.39}$$

It is then clear that all computations are similar at time t with an "updated" variance budget $\widetilde{\mathbb{V}} = \mathbb{V} - CV_t$ and using the values of V_t and S_t (instead of V_0 and S_0).

5.2 Numerical Analysis

We first show that our technique is accurate and fast using the Heston model because in the Heston model, we have a method that converges to the true price of a timer option. The tables are gathered at the end of this chapter.

5.2.1 Heston model

To simulate the price (5.20), we need to simulate (τ, V_τ). It is an almost "exact" simulation approach. The only bias comes from the simulation of (τ, V_τ) which is done using an Euler discretization of (5.22). Let $1/p$ be the discretization step for the Euler discretization needed to simulate (5.22) and n be the number of simulations.

In table 5.1, we give timer call prices for different levels of the correlation. We simulate (τ, V_τ) using (5.21) and (5.22). Then we use (5.25) when $\rho = 0$, and (5.20) when $\rho \neq 0$. Both approaches converge to the correct price when the number of simulations converge to $+\infty$ and the discretization step $1/p$ goes to 0.

Using remark 5.1.1, and specifically the closed-form expression given by equation (5.15), using the same parameters as in Panel A of table 5.1, we calculate the price of the timer call option, it is equal to 6.4871. It shows that our simulations are correct when $\rho = 0$. However we can note that the simulations when $\rho \neq 0$ have a higher standard deviation and depend on the discretization step used in the Euler discretization to simulate (τ, V_τ). When $r = 0\%$, remark 5.1.1 explains that the price should not depend on the correlation coefficient ρ. When the time step is 1/5000, the true price of the timer call option when $\rho = 0.8$ and $\rho = -0.8$ is 6.4871, and it lies in the confidence interval. However a discretization step of 1/500 is not enough to obtain an accurate result.

It is interesting to note that the approximation in table 5.2 is a very good approximation of the true price obtained in table 5.1. The standard deviations obtained by Antonelli and Scarlatti (2009) approximation are smaller than the ones obtained by (5.20) using Broadie and Kaya (2006)'s approach (in table 5.1 when the correlation ρ is not equal to 0). From tables 5.1 and 5.2, one can see that the approximation (5.30) is very good. In the case when the model is not Heston, a formula as (5.22) is not available and the approximation of Antonelli and Scarlatti (2009) is one way to get a quick and accurate price for the timer call option.

Hull and White Model

In table 5.3, we provide an example in the Hull and White setting. We note that the order of magnitude are similar. When the interest rate $r = 0\%$ the result is similar to the Heston case. This confirms the fact that the price of a timer option is model-free in this case. In the case when $r = 4\%$, the results between table 5.2 and 5.3 are quite different.

5.2.2 Impact of the Correlation Coefficient

The price of a timer option depends in a complicated way on the correlation coefficient between the underlying's asset price and its volatility. We first investigate how the price in our continuous setting given by (5.20) is sensitive to the correlation coefficient ρ. Then, we illustrate how this sensitivity can be quite different for discretely monitored timer options.

In this section, the setting is the Heston stochastic volatility model. Parameters we use are

$$\theta = 0.0324, \kappa = 2, \sigma_v = 0.1, V_0 = 0.0625, r = 4\%$$
$$S_0 = 100, \mathbb{V} = 0.0265, K = 100.$$

Continuous setting

We compare the price (5.20) obtained in the continuous setting with the approximation based on Antonelli and Scarlatti (2009) given by (5.30) and (5.31).

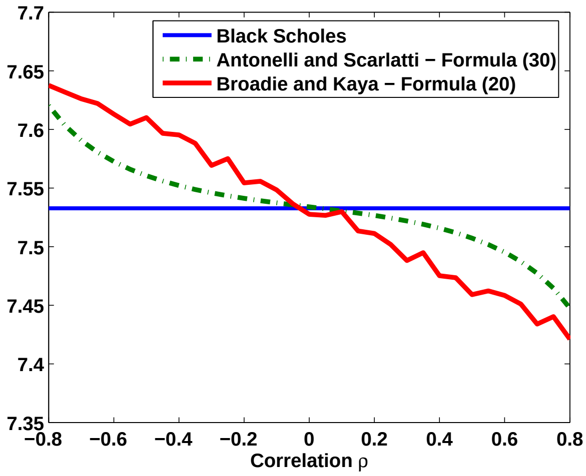

Figure 5.1: Timer Call Price w.r.t. ρ

With $S_0 = 100$, $K = 100$, $r = 4\%$, $V_0 = 0.0625$, $\kappa = 2$, $\theta = 0.0324$, $\sigma_v = 0.1$ (Heston model), $\mathbb{V} = 0.0265$ (variance budget). This is done with a time step of 1/3000 and 1, 000, 000 Monte Carlo simulations for each value of the correlation coefficient "ρ".

In Figure 5.1, both prices are displayed as a function of the correlation coefficient. They are computed with Monte Carlo simulations with 1, 000, 000 simulations and a time step of 1/3000. For the approximation by Antonelli and Scarlatti (2009), the result is obtained very quickly and it is very accurate (as could be seen from table 5.2). The corresponding curve on the graph is then very smooth. For the approximation based on (5.20), the standard deviation is quite high as can be seen from the corresponding curve in Figure 5.1. The Black and Scholes value is calculated with a maturity equal to $E_Q[\tau]$, a volatility $\sigma = \sqrt{\frac{\mathbb{V}}{T}}$, the same strike K, the initial price S_0 and the interest rate $r = 4\%$. Since the volatility is constant, it does not depend on the correlation coefficient ρ and we obtain an horizontal line as can be seen from Figure 5.1.

There are a few observations from Figure 5.1. First it shows that the approximation of Antonelli and Scarlatti (2009) works quite well to capture how the price of a timer option depends on the correlation parameter ρ.

Note that when the correlation is close to -1 or 1, the approximation is not very accurate.Recall that Antonelli and Scarlatti's approach is based on first order Taylor expansions. More terms are needed for a better approximation in the extreme cases of perfect correlation or anti-correlation. We can also observe that there is a dependence between the price of a timer option and the correlation coefficient. Note also that the price of a timer option is higher for negative correlation and lower for positive correlation. This is far from obvious and could easily be the contrary if the timer option is discretely monitored as it is discussed in the next paragraph. Finally the Black and Scholes formula can give an approximation of the price of the timer option when its correlation is equal to zero as can also be seen from the Figure 5.1.

Discretely monitored timer options

We proceed by a direct simulation of the *joint* correlated processes (S_t, V_t) to obtain 500000 simulations of (S_τ, τ) with a discretization step $\Delta_t = 1/M$ where M varies. Then we calculate the price of the timer call option by crude Monte Carlo using

$$E^Q \left[e^{-r\tau} \max\left(S_\tau - K, 0\right) \right]. \tag{5.40}$$

The graph for the sensitivity of timer call price with M is given below.

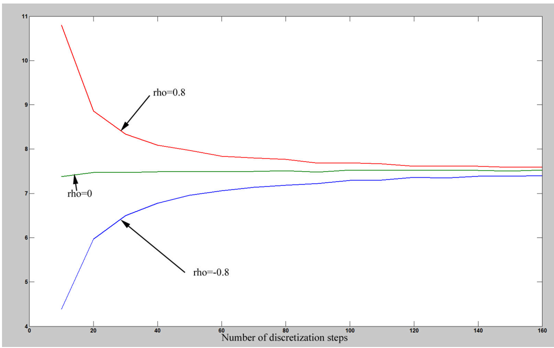

Figure 5.2: Timer Call Price w.r.t. M, the number of discretization steps With $S_0 = 100$, $K = 100$, $r = 4\%$, $V_0 = 0.0625$, $\kappa = 2$, $\theta = 0.0324$, $\sigma_v = 0.1$ (Heston model), $\mathbb{V} = 0.0265$ (variance budget).

The price obtained by this first method converges to the correct price of the timer call option when the number of simulations M goes to $+\infty$ and when the discretization step goes to 0. Results obtained with this first approach are displayed in Figure 5.2. We fix the number of steps M and simulate 500000 simulations of (τ, S_τ). With the same random numbers we calculate (5.40) for three possible correlations $\rho = 0$, $\rho = 0.8$ and $\rho = -0.8$. Then we change M and use new random numbers to calculate (5.40) by Monte Carlo.

From Figure 5.2, it seems that the price of the timer call option is not very sensitive to the correlation parameter when the discretization step is about 1/200, which represents almost a trading day. At the right end of the graph on Figure 5.2, the price of a call option with $\rho = 0.8$ is 7.60, with $\rho = 0$, it is 7.53 and with $\rho = -0.8$ it is 7.40. Figure 5.2 confirms that the price of the timer call weakly depends on the correlation when the discretization step gets small (in particular in the case of continuous monitoring). However, it can also be seen from the graph that this dependence on correlation is very important for a monthly time step ($M = 12$) or for a weekly time step ($M = 52$). Approximating the price of the timer option by the price of it when $\rho = 0$ can be a good approximation when the realized variance is continuously monitored (or at least monitored daily, with $M \geqslant 250$).

We went into more details to study the actual dependence with respect to the correlation coefficient ρ by using a thinner time step. We used a discretization step as small as 1/7000 year (which roughly corresponds to half an hour assuming 252 trading days with 12 hours per day). We found that the price of the timer call option is respectively 7.63 when $\rho = 0.8$, 7.53 when $\rho = 0$ and 7.43 when $\rho = -0.8$ which is consistent with table 5.1.

Remark 5.2.1. *When there is a large discretization step (when M is small) the timer call is more valuable when the correlation is positive (see Figure 5.2). But when the discretization step is very thin, the contrary can be observed and the timer call is more valuable when the correlation is negative. This also confirms the result obtained in the continuous case* *displayed in Figure 5.1.*

Remark 5.2.1 is very important because it shows that one should be very careful when evaluating timer options in practice. Indeed stock processes are usually discretely monitored in practice. Therefore, in the continuous-time setting, the sensitivity of the timer call to the correlation coefficient might be contrary to what one usually expects.

5.3 Practical implications for this design

We first discuss standard timer call options and their practical interest. Then we present the capped timer options (discussed by Lehman Brothers (2008)) and two more recent timer-style contracts that SG CIB started to sell the "timer out-performance option" and the "time swap".

5.3.1 Standard Timer Options

We already mentioned in the introduction and in section 5.1.1 some of the practical interests of trading timer options. The first one is to be able to buy call options at a cheaper price than standard vanilla options since the realized volatility is often lower than the implied volatility, and the price of a standard option is determined by the level of implied volatility. By taking a long position in a timer option rather than in a standard option, the investor is exposed to uncertainty about the maturity of the contract. It will be exactly equal to the target investment horizon of the contract if the volatility target chosen by the investor at inception of the contract matches exactly the realized volatility. *"High implied volatility means call options are often overpriced. In the timer option, the investor only pays the real cost of the call and does not suffer from high implied volatility, "* says Stephane Mattatia, head of the hedge fund engineering team at SG CIB in Paris.

The Timer option is an example of quadratic-variation-based derivative. We can expect that quadratic-variation-based strategies will be useful to hedge timer options as well as timer options can be useful for implementing quadratic-variation-based strategies. For example section 4 of Geman and Yor (1993) is dedicated to quadratic-variation-based strategies. They explain that "Rendleman and O'Brien (1990) have shown that misspecification of the volatility can cause the outcome of a synthetic portfolio insurance strategy to deviate significantly from its target." Therefore it is important to consider new approaches. A very innovating and interesting approach was proposed by Bick (1995). The idea is to develop a portfolio insurance strategy with a random maturity time by proposing to stop the strategy when the realized variance of the portfolio hits some prespecified level. This is the same spirit as the timer option. At the time it was proposed by Bick (1995), these options did not exist. See also Geman (2008). Note that Bick (1995) shows that timer options admit perfect replication by dynamically trading in the underlying risky asset and zero-coupon bonds.

5.3.2 Existing exotic timer options

Lehman Brothers (2008) discussed capped timer options. Société Générale already started to sell several timer-style options FX timer options, time swaps and timer out-performance options. In this section, we show how our study can be extended to these options, and how approximations for their prices can be derived as well. For each product, we provide a description of the product, a pricing formula and propose some applications.

Capped Timer-type products

In practice, timer options are often proposed with a maximum horizon (See Lehman Brothers (2008)). Investors may be reluctant to invest in timer options because of the too high uncertainty about the time horizon. The major fear can be that the maturity of the option is very long and far later than the actual horizon of the manager. In this section, we propose to cap the maturity date at a fixed maturity date T. Let us illustrate this concept on a "capped timer call option". The investor specifies two parameters, his maximum investment horizon T and a variance budget $\mathbb{V}$. A capped timer call option expires at $\min(\tau, T)$, where

$$\tau = \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}\right\}.$$

Its price in the standard framework can be given as

$$E_Q[e^{-r\tau}\mathbb{1}_{\tau<T}(S_\tau - K)^+ + e^{-rT}\mathbb{1}_{\tau\geqslant T}(S_T - K)^+].$$

Numerical techniques are now needed to calculate this price. Methods are similar to the case of standard timer options.

FX Timer options

A FX timer option is very similar to a timer call. The only difference lies in the underlying process. In a FX timer option, the investor sets a variance budget for the exchange rate and it is a call option on the exchange rate (it is a "timer caplet"). When the accumulated variance of the exchange rate has reached the budget, the caplet expires. Prices are derived such as in section 5.1.3 and practical interests are similar to standard timer options. More about modeling FX rates can be found in Wystup (2007).

Time swap

A word description of a "time swap" and why it can be useful can be found in Sawyer (2008). *"The time swap, on the other hand, gives investors a means to short volatility with an inverted convexity profile (meaning the downside is limited). Rather than volatility, the strike is expressed in days. In other words, if an investor wants to short volatility over a specified investment period - for instance, three months - a variance budget is calculated with a volatility level set by SG CIB. The payout is based on the number of days required to consume the variance budget minus the specified investment horizon, times the notional. So, if realised volatility is consistently lower than the specified level, it will take longer than three months for the option to expire, and the investor receives a payout."*

Let us now formalize this definition and show how to price a time swap. The idea is to compare the target expiry time T with the random time τ

defined as before by $\tau = \inf\{u > 0, \int_0^u V_s ds = \mathbb{V}\}$. As a standard swap, a time swap has a notional amount, K. At the time τ when the variance budget is expended, the payoff of the time swap is

$$K(\tau - T). \tag{5.41}$$

Then, the price of the time swap is given as follows

$$KE\left[e^{-r\tau}(\tau - T)\right]. \tag{5.42}$$

This contract allows investors to take a short position to realized volatility with a limited risk exposure. The payoff is indeed positive if τ occurs after T and is negative when τ occurs prior to T. The investor will therefore receive money when the realized volatility is smaller than the target, and lose money otherwise. This position is then a short position with respect to the realized volatility. However, unlike selling a call option for instance, it has limited exposure in the sense that $\tau \geqslant 0$ and therefore the maximum loss is KT (as can be seen from the payoff (5.41)). It might therefore be a useful hedging tool for the timer call option.

As soon as we know how to simulate in an accurate and fast way the random time τ, it is easy to estimate this price by Monte Carlo from (5.42). Note that unlike the timer call option, the price of the time swap does not involve the correlation ρ.

Timer out-performance option

This section extends some results from a presentation by Li (2008) we propose an alternative pricing approach and slightly different contract. The timer out-performance option was developed shortly after the first timer call option was sold in April 2007. Sawyer (2008) explains that *"the out-performance product is similar to the timer call the investor specifies a target volatility for the spread between two underlyings and a target investment horizon, which is used to calculate a variance budget. Mattatia claims the timer out-performance call can be 30 percent cheaper than a plain vanilla out-performance option - that's because the price of an out-performance* *option depends on the implied volatility levels of both underlyings (meaning the investor is overpaying volatility) and implied correlation between the two (which is usually under-priced).*

To model a timer out-performance option, one needs to introduce two correlated assets S_1 and S_2. The investor then specifies a target volatility σ_0 for the spread between the two underlying's log-return and a target investment horizon T. A variance budget is calculated as

$$\mathbb{V} = \sigma_0^2 T.$$

Define τ as the first time the accumulated variance of the spread between the log-returns of the two assets reaches $\mathbb{V}$. The payoff at time τ of the timer out-performance option can then be simply expressed as

$$\max\left(S_2(\tau) - S_1(\tau), 0\right).$$

This option could also be called "Timer Exchange Option". In the special case when the two assets share the same underlying volatility, a simple formula can be obtained.

Let us first model the problem, then the price is given in Proposition 5.3.1. We first define the realized variance RV_u of the log-return spreads between the two correlated assets at time u. Consider n dates $0 \leqslant t_1 < t_2 < ... < t_n \leqslant T$, the realized variance is calculated as

$$RV_T(n) = \sum_{i=1}^{n-1} \left(\ln\left(\frac{S_1(t_{i+1})}{S_1(t_i)}\right) - \ln\left(\frac{S_2(t_{i+1})}{S_2(t_i)}\right)\right)^2. \tag{5.43}$$

It can be also written as

$$RV_T(n) \quad = \quad \sum_{i=1}^{n-1} \left(\ln\left(\frac{S_2(t_{i+1})}{S_1(t_{i+1})}\right) - \ln\left(\frac{S_2(t_i)}{S_1(t_i)}\right)\right)^2. \tag{5.44}$$

From (5.44), and by definition of the quadratic-variation,

$$\lim_{n\to\infty} RV_T(n) = \left\langle \ln\left(\frac{S_2}{S_1}\right)\right\rangle_T \tag{5.45}$$

Assume that the two assets and their common variance evolve with the following dynamics under the risk neutral measure Q,

$$
\begin{aligned}
dS_1(t) &= rS_1(t)dt + \sqrt{V_t}S_1(t)dW_1(t) \\
dS_2(t) &= rS_2(t)dt + \sqrt{V_t}S_2(t)dW_2(t) \\
dV_t &= \alpha(V_t)dt + \beta(V_t)dW_0(t)
\end{aligned}
\tag{5.46}
$$

where the standard Brownian motions W_1, W_2 and W_3 are correlated such that

$$
\begin{cases}
dW_0(t) \cdot dW_1(t) &= \rho_{01}dt \\
dW_0(t) \cdot dW_2(t) &= \rho_{02}dt \\
dW_1(t) \cdot dW_2(t) &= \rho_{12}dt
\end{cases}
$$

and where ρ_{01}, ρ_{02}, ρ_{12} are constant. Define then

$$
\tau = \inf\left\{u > 0, \left\langle \ln \frac{S_2}{S_1} \right\rangle_u = \mathbb{V}\right\}.
$$

The payoff at time τ of the timer out-performance option is given by

$$
\max\left(S_2(\tau) - S_1(\tau), 0\right).
$$

Proposition 5.3.1. *The price of a timer out-performance option is given by*

$$
M_0 = S_2(0)\mathcal{N}(d_1) - S_1(0)\mathcal{N}(d_2) \tag{5.47}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(S_2(0)/S_1(0)\right) + \frac{\mathbb{V}}{2}}{\sqrt{\mathbb{V}}}, \\
d_2 &= d_1 - \frac{\sqrt{\mathbb{V}}}{2}.
\end{aligned}
\tag{5.48}
$$

The proof of Proposition 5.3.1 is given in Appendix 5.7. The explicit expression given in Proposition 5.3.1 shows that the price of this timer out-performance option does not depend on the form of the dynamics of the variance process. This means that the explicit price formula works for very general stochastic volatility models with a general correlation structure given by (5.47).

It is also possible to consider a timer exchange option between two assets that have different volatilities. In this case there is no simple expression.

Exchange options are used when the investor has a view of the comparative performance of two assets. This timer exchange option is useful when investor knows that some asset price is more affected to the market volatility fluctuation than the other asset. If the investor judges that as the volatility increases to a certain level, asset 1 will increase more than asset 2 does, then he can enter into a timer exchange option to exchange asset 2 for asset 1.

We can expect that more exotic timer-style options will be issued in the future since it might be a way of hedging volatility risk at a lower cost. We now propose some possible future designs.

5.4 Proposal for More Exotic Timer Options

In this section we propose to design a wide variety of timer-style exotic derivative products. As far as we know, the products that are described and priced in this section are not traded yet. In the future they might be traded because of their interesting features. In fact, any path-independent European option can be evaluated along the lines for pricing the standard timer option described in section 5.1.3. It is also possible to design "timer-style" mild path-dependent options, such as forward start options and consequently Ratchet options (also called Cliquet options) or compound options as well as highly path-dependent options. Note that the report by Lehman Brothers (2008) concludes by pointing out that *"the further study of combinations of Timer Options with Vanilla Options should offer some interesting opportunities"*.

5.4.1 Forward start timer option

A forward start time option has the feature of a "forward start" option. At time 0, the investor specifies two variance budgets $\mathbb{V}_1$ and $\mathbb{V}_2$, with $\mathbb{V}_1 < \mathbb{V}_2$. The Forward start timer call option starts when the accumulated variance reaches the variance budget $\mathbb{V}_1$ and expires when it reaches the budget $\mathbb{V}_2$. Let us now formalize the problem. We define τ_1 and τ_2 as follows

$$\begin{aligned} \tau_1 &= \inf \left\{ u > 0, \int_0^u V_s ds = \mathbb{V}_1 \right\} \\ \tau_2 &= \inf \left\{ u > 0, \int_0^u V_s ds = \mathbb{V}_2 \right\} \end{aligned}$$

Since $\mathbb{V}_1 < \mathbb{V}_2$ and the accumulated variance is an increasing process, $\tau_1 < \tau_2$. The payoff of a forward start timer option is related to the return on the asset between τ_1 and τ_2. It could be linked to the ratio $\frac{S_{\tau_2}}{S_{\tau_1}}$ or to a weighted difference between S_{τ_1} and S_{τ_2}.

The payoff of the ratio forward start call option at expiry time τ_2 could be

$$\max \left(\frac{S_{\tau_2}}{S_{\tau_1}} - k, 0 \right), \tag{5.49}$$

where k is the strike at τ_2. This payoff has the advantage to be very similar to the case of a standard timer option. From expression (5.11) we know that

$$S_{\tau_1} = S_0 e^{r\tau_1} e^{B_{\mathbb{V}_1} - \frac{1}{2}\mathbb{V}_1} \text{ and } S_{\tau_2} = S_0 e^{r\tau_2} e^{B_{\mathbb{V}_2} - \frac{1}{2}\mathbb{V}_2}. \tag{5.50}$$

Then,

$$\frac{S_{\tau_2}}{S_{\tau_1}} = e^{r(\tau_2 - \tau_1)} e^{B_{\mathbb{V}_2} - B_{\mathbb{V}_1} - \frac{1}{2}(\mathbb{V}_2 - \mathbb{V}_1)}. \tag{5.51}$$

In the case when $\rho = 0$, $B_{\mathbb{V}_2} - B_{\mathbb{V}_1}$ is independent of τ_1 and τ_2, and

$$B_{\mathbb{V}_2} - B_{\mathbb{V}_1} - \frac{1}{2}(\mathbb{V}_2 - \mathbb{V}_1) \sim \mathcal{N}\left(-\frac{1}{2}(\mathbb{V}_2 - \mathbb{V}_1), \mathbb{V}_2 - \mathbb{V}_1 \right). \tag{5.52}$$

Values of τ_1 and τ_2 can be simulated by the methods described in the first section, and the price C_0 of the ratio forward start timer call option can be approximated by Monte Carlo

$$C_0 = E_Q\left[e^{-r\tau_1}\mathcal{N}(d_1) - ke^{-r\tau_2}\mathcal{N}(d_2)\right], \tag{5.53}$$

where $d_1 = \frac{\frac{1}{2}(\mathbb{V}_2-\mathbb{V}_1)-\ln\left(ke^{-r(\tau_2-\tau_1)}\right)}{\sqrt{\mathbb{V}_2-\mathbb{V}_1}}$ and $d_2 = d_1 - \sqrt{\mathbb{V}_2 - \mathbb{V}_1}$. The ratio forward start timer *put* option price can be obtained similarly.

The payoff of the option at expiry time τ_2 can also be

$$\max\left(S_{\tau_2} - kS_{\tau_1}, 0\right), \tag{5.54}$$

where k is the fraction of the old stock price that will be reset as the new strike.

The options (5.49) and (5.54) can be tailor-made to investors with different volatility preferences. The lower variance budget $\mathbb{V}_1$ is a minimum accumulated variance for the investor to enter into the option contract. The upper bound $\mathbb{V}_2$ is the maximum accumulated variance that the investor could tolerate before the exercise of the option. It seems that it could have interesting practical implications since the investors will be able to quantify the volatility risk they want to undertake. This product is very similar to European forward start options, but here the benchmark are variance budgets rather than the time to maturity. In the classical case, the investor can only control the specified times T_1 and T_2 and will suffer from uncertain volatility risk. Using this timer style option, it may help to control the maximum acceptable volatility risk of the investor.

Timer Cliquet Options

A cliquet option consists of a portfolio of comsecutive forward start options. Therefore we may as well design Cliquet timer-style options. Thus the "timer cliquet option" with n resetting periods is a portfolio of n forward start timer options. In this contract, the investor needs to specify n

variance budgets at time $0 < \mathbb{V}_1 < \mathbb{V}_2 < ... < \mathbb{V}_n$. Let

$$\begin{aligned}
\tau_1 &= \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}_1\right\} \\
\tau_2 &= \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}_2\right\} \\
&\dots \\
\tau_n &= \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}_n\right\}
\end{aligned}$$

A cliquet option can be designed using a portfolio of $n-1$ forward start timer options with respective variance budgets, $\mathbb{V}_{i-1}$ and $\mathbb{V}_i$ for $i = 2..n$.

5.4.2 Compound Timer Option

Assume the investor specifies two variance budget levels $\mathbb{V}_1 < \mathbb{V}_2$ and two strikes K_1 and K_2. We define

$$\tau_1 = \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}_1\right\}, \ \tau_2 = \inf\left\{u > 0, \int_0^u V_s ds = \mathbb{V}_2\right\}.$$

It is obvious that τ_2 will happen after τ_1 since $\mathbb{V}_2 > \mathbb{V}_1$. Let us now illustrate the timer compound option with a timer call on a timer call, other cases can be similarly derived. At time τ_1, the holder of the contract has the right to buy at the strike price K_1, a timer call option. This latter timer call option gives the holder the right to buy the underlying asset at the strike price K_2 at time τ_2. So the payoff of the option at τ_1 is

$$\max\left(C\left(\tau_1, \tau_2, K_2\right) - K_1, 0\right)$$

where $C\left(\tau_1, \tau_2, K_2\right)$ is the price at time τ_1 of the second timer call option starting from time τ_1 and expiring at time τ_2 with maturity payoff $\max\left(S_{\tau_2} - K_2, 0\right)$ [2].

A particularly interesting compound option may be the timer call on the timer put. It is especially interesting if one anticipates to buy a put option in the future to hedge a guarantee for instance. One may need it

[2] In the case when $\rho = 0$, following Geske (1977), formulas can be obtained for the compound timer options and are available from me upon request.

when the volatility in the market gets very high or after a few jumps in the stock price. But at that time the put option can become very expensive. It might therefore be useful to enter in a timer call on a timer put option (or even written on a standard put option if the guarantee is at a fixed date).

5.4.3 Highly path-dependent timer-style options

While the maturity date is fixed and the accumulated variance is random for a standard call option, the contrary holds for the timer option. The key feature in a timer-style option is to have a random maturity but a fixed volatility exposure. In this last section, we propose to extend discretely monitored options such as discrete lookback options, discrete barrier options, discrete Asian options to new timer-style derivatives. Instead of equally spaced time intervals, one has intervals linked to the accumulated variance process.

Let us specify a variance budget $\mathbb{V}$, and assume that the financial derivative expires at time τ when the accumulated variance of an underlying asset reaches $\mathbb{V}$. Now let us divide the interval $[0, \mathbb{V}]$ into n equally spaced intervals as follows $0 < \frac{\mathbb{V}}{n} < \frac{2\mathbb{V}}{n} < \frac{3\mathbb{V}}{n} < ... < \frac{(n-1)\mathbb{V}}{n} < \mathbb{V}$. Define the hitting times of the realized variance process to each of these intermediate levels respectively as $\tau_1, \tau_2, ..., \tau_{n-1}$ and τ_n is then equal to the maturity of the contract, $\tau_n = \tau$. Then, $\tau_1 ... \tau_n$ are sampling dates.

With the timer-style options, the number of observations of the process will increase when the volatility is high and will decrease when the volatility is low. This is an interesting feature for the discrete lookback timer options, the discrete barrier timer options, and the discrete Asian timer options that we present below.

Discrete Lookback Timer options

We now record respectively the maximum and the minimum asset value at these stopping times, respectively denoted by $M(\mathbb{V}, n)$ or $m(\mathbb{V}, n)$.

$$M(\mathbb{V}, n) = \max_{i=1..n} \{S(\tau_i)\}\,,\; m(\mathbb{V}, n) = \min_{i=1..n} \{S(\tau_i)\}\,,$$

The payoff of a discrete lookback timer call is defined as $\max\left(M(\mathbb{V}, n) - K, 0\right)$, and the payoff of a timer lookback put option payoff is $\max\left(K - m(\mathbb{V}, n), 0\right)$.

A discrete lookback timer option shares the same structure as a standard discretely monitored Lookback option. However it may take advantage of changes in the volatility over the life of the contract. In periods of high volatility, there are more obervations of the underlying's process. Thus this contract may be closer to the continuously monitored lookback options.

Discrete Barrier Timer option

A discrete barrier timer option is similar to a standard discrete barrier option. The only difference lies in the observation dates. In the timer style barrier option, the underlying is observed on the specified sampling dates τ_i rather than on equally-spaced dates to check if the asset price crosses some specific threshold. For example, in the case of a standard down and in put option, it is activated if the underlying hits a low barrier before maturity. If the process is discretely monitored, say each week and that during a particular week there is a lot of volatility, the underlying asset may hit the level but may be above at the end of the week. With the timer-style barrier option, the number of observations of the process will increase when the volatility is high and will decrease when the volatility is low.

Discrete Asian Timer option

Discrete geometric Asian timer option or discrete arithmetic Asian timer option will involve the calculation of the average of $S(\tau_i), i = 1..n$ instead of the average of equally-spaced observations.

In the case of the geometric average, and when $\rho = 0$ a closed-form expression can be derived. We know already that

$$S_{\tau_i} = S_0 e^{r\tau_i} e^{B_{\xi(\tau_i)} - \frac{1}{2}\xi(\tau_i)} = S_0 e^{B_{\mathbb{V}_i} - \frac{1}{2}\mathbb{V}_i} e^{r\tau_i}, \tag{5.55}$$

for $i = 1, 2, ..., n$ with $\mathbb{V}_i = \frac{i\mathbb{V}}{n}$. Thus the price of the discrete geometric

Asian timer call is given by

$$C_{Geo} = E_Q \left[S_0 e^{r(A_n - \tau) + \frac{1}{2}(\sigma_n - B_n)} \mathcal{N}(d_1) - K e^{-r\tau} \mathcal{N}(d_2) \right] \tag{5.56}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + rA_n - \frac{1}{2}B_n + \sigma_n^2}{\sigma_n}, \\ d_2 &= d_1 - \sigma_n. \end{aligned} \tag{5.57}$$

and

$$\begin{aligned} \sigma_n^2 &= \frac{(n+1)(2n+1)}{6n^2}\mathbb{V}, \\ A_n &= \frac{\tau_1 + \tau_2 + ... + \tau_{n-1} + \tau_n}{n}, \\ B_n &= \frac{(n+1)}{2n}\mathbb{V}. \end{aligned} \tag{5.58}$$

In addition the average of the underlying asset will involve more observations in the period where there is more volatility, and fewer observations in the periods when the volatility is lower. The average will better reflect the extreme values of the process if any.

5.5 Proof of formula (5.18)

In the case of the Heston stochastic volatility model, we can solve the problem. Following the methodology discussed by Broadie and Kaya (2006), we integrate both sides of the equation (5.5) from 0 to τ. Then we obtain

$$V_\tau = V_0 + \kappa\theta\tau - \kappa \int_0^\tau V_s ds + \sigma_v \int_0^\tau \sqrt{V_s} dW_s^2. \tag{5.59}$$

Note that $\int_0^\tau V_s ds = \mathbb{V}$ by definition of τ. Therefore,

$$V_\tau = V_0 + \kappa\theta\tau - \kappa\mathbb{V} + \sigma_v \int_0^\tau \sqrt{V_s} dW_s^2 \tag{5.60}$$

and the stochastic integral term in the above equation can be expressed as

$$B^2_{\xi(\tau)} = B^2_{\mathbb{V}} = \int_0^\tau \sqrt{V_s} dW_s^2 = \frac{V_\tau - V_0 - \kappa\theta\tau + \kappa\mathbb{V}}{\sigma_v} \tag{5.61}$$

Given the values of τ and V_τ, we can see that the above quantity is a constant.

5.6 Proof of (5.26) in the Hull and White Model

Recall the Hull and White stochastic volatility model under the risk neutral measure Q.

$$\begin{aligned} dS_t &= rS_t dt + S_t\sqrt{V_t}\left(\rho dW_t^2 + \sqrt{1-\rho^2}dW_t^3\right) \\ dV_t &= \mu_v V_t dt + \xi_v V_t dW_t^2 \end{aligned} \tag{5.62}$$

Let us consider the following function f

$$f(x) = \int_0^x \frac{1}{\xi_v\sqrt{z}}dz = \frac{2}{\xi_v}\sqrt{x} \tag{5.63}$$

By Ito's lemma, we have

$$df(V_t) = \left(\frac{\mu_v}{\xi_v} - \frac{\xi_v}{4}\right)\sqrt{V_t}dt + \sqrt{V_t}dW_t^2 \tag{5.64}$$

Now we integrate both sides of (5.64) from 0 to τ, and obtain

$$\int_0^\tau \sqrt{V_t}dW_t^2 = \frac{2}{\xi_v}\sqrt{V_\tau} - \frac{2}{\xi_v}\sqrt{V_0} - \int_0^\tau \left(\frac{\mu_v}{\xi_v} - \frac{\xi_v}{4}\right)\sqrt{V_t}dt \tag{5.65}$$

Following similar ideas as in the proof of proposition 3 of Li [2009b], define $\tau_t = \inf\{u > 0, \int_0^u V_s ds = t\}$, then by Dubins-Schwartz theorem, we have

$$\int_0^{\tau_t} \sqrt{V_s}dW_s^2 = B_t. \tag{5.66}$$

We also have $\tau_t = \int_0^t \frac{1}{V(\tau_s)}ds$ and $\tau_t' = \frac{1}{V(\tau_t)}$. From (5.65) and (5.66), we have

$$\frac{2}{\xi_v}\left(\sqrt{V_{\tau_t}} - \sqrt{V_0}\right) = \left(\frac{\mu_v}{\xi_v} - \frac{\xi_v}{4}\right)\int_0^{\tau_t} \sqrt{V_s}ds + B_t \tag{5.67}$$

Take derivative with respect to t on both sides of (5.67)

$$
\begin{aligned}
d\sqrt{V_{\tau_t}} &= \left(\frac{\mu_v}{2} - \frac{\xi_v^2}{8}\right)\sqrt{V_{\tau_t}}\tau_t^{'}dt + \frac{\xi_v}{2}dB_t. \\
&= \left(\frac{\mu_v}{2} - \frac{\xi_v^2}{8}\right)\frac{1}{\sqrt{V_{\tau_t}}}dt + \frac{\xi_v}{2}dB_t
\end{aligned} \tag{5.68}
$$

Now we let $X_t = \frac{2}{\xi_v}\sqrt{V_{\tau_t}}$, write it in another form

$$
dX_t = \left(\frac{2\mu_v}{\xi_v^2} - \frac{1}{2}\right)\frac{1}{X_t}dt + dB_t, \quad X_0 = \frac{2}{\xi_v}\sqrt{V_0}. \tag{5.69}
$$

Since

$$
\begin{aligned}
V_\tau &= V_{\tau_{\mathbb{V}}} = \frac{\xi_v^2}{4}X_{\mathbb{V}}^2 \\
\tau &= \tau_{\mathbb{V}} = \int_0^{\mathbb{V}} \frac{1}{V_{\tau_s}}ds = \frac{4}{\xi_v^2}\int_0^{\mathbb{V}} \frac{1}{X_s^2}ds,
\end{aligned} \tag{5.70}
$$

it is possible to simulate (τ, V_τ) jointly from the simulation of X_t.

For other general stochastic volatility models, similar results can be obtained by choosing an appropriate function f in (5.63), and derive the dynamic for X_t, needed to simulate (τ, V_τ).

5.7 Out-performance Options

Using Cholesky decomposition, we can decompose the three correlated Brownian motions in equations (5.46) into three independent standard Q-Brownian motions B_i, $i = 0, 1, 2$.

$$
\begin{cases}
dW_0(t) = dB_0(t) \\
dW_1(t) = \rho_{01}dB_0(t) + \alpha_{11}dB_1(t) \\
dW_2(t) = \rho_{02}dB_0(t) + \alpha_{12}dB_1(t) + \alpha_{22}dB_2(t),
\end{cases} \tag{5.71}
$$

where $\alpha_{11} = \sqrt{1-\rho_{01}^2}, \alpha_{12} = \frac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{1-\rho_{01}^2}}, \alpha_{22} = \sqrt{1-\rho_{02}^2 - \frac{(\rho_{12}-\rho_{01}\rho_{02})^2}{1-\rho_{01}^2}}$.

By Ito's lemma, under the risk-neutral measure Q, we have

$$
\begin{aligned}
d\left(\frac{S_2(t)}{S_1(t)}\right) &= \left(\frac{S_2(t)}{S_1(t)}\right)\sqrt{V_t}(dW_2(t) - dW_1(t)) \\
&= \left(\frac{S_2(t)}{S_1(t)}\right)\sqrt{V_t}\left((\rho_{02}-\rho_{01})dB_0(t) + (\alpha_{12}-\alpha_{11})dB_1(t) + \alpha_{22}dB_2(t)\right)
\end{aligned} \tag{5.72}
$$

Then

$$\left\langle \ln \frac{S_2}{S_1} \right\rangle_u = \left[(\rho_{02}-\rho_{01})^2 + (\alpha_{12}-\alpha_{11})^2 + \alpha_{22}^2\right] \int_0^u V_s ds$$
$$= (2-2\rho_{12}) \int_0^u V_s ds \qquad (5.73)$$

Define then

$$\tau = \inf\left\{u>0, \left\langle \ln \frac{S_2}{S_1} \right\rangle_u = \mathbb{V}\right\}$$
$$= \inf\left\{u>0, (2-2\rho_{12}) \int_0^u V_s ds = \mathbb{V}\right\}$$
$$= \inf\left\{u>0, \int_0^u V_s ds = \overline{\mathbb{V}}\right\}. \qquad (5.74)$$

where $\overline{\mathbb{V}} = \frac{\mathbb{V}}{2-2\rho_{12}}$. The payoff at time τ of the timer out-performance option is given by

$$\max\left(S_2(\tau) - S_1(\tau), 0\right).$$

From equation (5.46),

$$S_1(\tau) = S_1(0) e^{r\tau} e^{N_1(\tau) - \frac{1}{2}\overline{\mathbb{V}}}$$
$$S_2(\tau) = S_2(0) e^{r\tau} e^{N_2(\tau) - \frac{1}{2}\overline{\mathbb{V}}} \qquad (5.75)$$

where $N_1(\tau) = \rho_{01}\int_0^\tau \sqrt{V_t} dB_0(t) + \alpha_{11}\int_0^\tau \sqrt{V_t} dB_1(t)$, and $N_2(\tau) = \rho_{02}\int_0^\tau \sqrt{V_t} dB_0(t) + \alpha_{12}\int_0^\tau \sqrt{V_t} dB_1(t) + \alpha_{22}\int_0^\tau \sqrt{V_t} dB_2(t)$.

We can calculate the price at time 0 of this timer out-performance option, denoted by M_0, as follows

$$M_0 = E^Q\left[e^{-r\tau}\left(S_2(\tau) - S_1(\tau)\right)^+\right]$$
$$= E^Q\left[e^{\rho_{01}\int_0^\tau \sqrt{V_t} dB_0(t) + \alpha_{11}\int_0^\tau \sqrt{V_t} dB_1(t) - \frac{1}{2}\overline{\mathbb{V}}} \left(S_2(0) e^{R(\tau)} - S_1(0)\right)^+\right] \qquad (5.76)$$

where

$$R(\tau) = (\rho_{02}-\rho_{01}) \int_0^\tau \sqrt{V_t} dB_0(t) + (\alpha_{12}-\alpha_{11}) \int_0^\tau \sqrt{V_t} dB_1(t) + \alpha_{22} \int_0^\tau \sqrt{V_t} dB_2(t)$$

Now we change measure from Q to Q^1 with the following Radon-Nikodym derivative

$$\frac{dQ^1}{dQ}(t) = \exp\left\{\rho_{01}\int_0^t \sqrt{V_u} dB_0(u) + \alpha_{11}\int_0^t \sqrt{V_u} dB_1(u) - \frac{1}{2}\int_0^t V_u du\right\} \qquad (5.77)$$

because $\rho_{01}^2 + \alpha_{11}^2 = 1$.

By the Multidimensional Girsanov Theorem, we have the following three independent Brownian motions under Q^1

$$\begin{aligned} d\tilde{B}_0(t) &= dB_0(t) - \rho_{01}\sqrt{V_t}dt \\ d\tilde{B}_1(t) &= dB_1(t) - \alpha_{11}\sqrt{V_t}dt \\ d\tilde{B}_2(t) &= dB_2(t) \end{aligned} \tag{5.78}$$

Then we can rewrite $R(\tau)$ under the new measure Q^1

$$\begin{aligned} R(\tau) &= (\rho_{02} - \rho_{01}) \int_0^\tau \sqrt{V_t}d\tilde{B}_0(t) + (\alpha_{12} - \alpha_{11}) \int_0^\tau \sqrt{V_t}d\tilde{B}_1(t) + \alpha_{22} \int_0^\tau \sqrt{V_t}d\tilde{B}_2(t) \\ &\quad -(1 - \rho_{12}) \int_0^\tau V_u du \end{aligned} \tag{5.79}$$

Now by Dubins-Schwartz theorem, $\int_0^\tau \sqrt{V_t}d\tilde{B}_i(t) = \tilde{Z}_i(\overline{\mathbb{V}})$ for $i = 0, 1, 2$ and these are three independent Brownian motions under Q^1 with variance $\overline{\mathbb{V}}$. Then using equation (5.76), one obtains

$$\begin{aligned} M_0 &= E^Q\left[e^{\rho_{01}\int_0^\tau \sqrt{V_t}dB_0(t) + \alpha_{11}\int_0^\tau \sqrt{V_t}dB_1(t) - \frac{\overline{\mathbb{V}}}{2}} \left(S_2(0)e^{R(\tau)} - S_1(0)\right)^+\right] \\ &= E^{Q^1}\left[\left(S_2(0)e^{R(\tau)} - S_1(0)\right)^+\right] \end{aligned} \tag{5.80}$$

Now from equation (5.79),

$$R(\tau) \sim \mathcal{N}\left(-\left(1 - \rho_{12}\right)\overline{\mathbb{V}}, \mathbb{V}\right) = \mathcal{N}\left(-\frac{\mathbb{V}}{2}, \mathbb{V}\right) \tag{5.81}$$

So we can easily calculate the expectation term and prove Proposition 5.3.1.

5.8 Conclusion of Chapter 5

We have proposed an accurate and fast technique to approximate the price of a timer option in various stochastic volatility models. The cases of the Heston and Hull and White stochastic volatility models are used for the purpose of illustration. However it is clear that the approach of Antonelli and Scarlatti (2009) can be extended to many other stochastic volatility models. When the correlation is close to 1 or -1, first order terms given by the method of Antonelli and Scarlatti (2009) are not very precise and more Taylor expansion terms are needed.

Because of the booming of derivative markets for variance of stock returns (see Carr and Lee (2009), Carr and Madan (1998)), such as variance swaps, volatility swaps, corridor gamma swaps, the market for volatility products is more and more important. We expect that most of the contracts proposed in this paper may become available, and that advanced hedging strategies will be derived from the use of such volatility derivatives. These timer products may probably serve as hedging or replication tools for variance swaps or volatility swaps or vice versa and help to complete the market.

Table 5.1: Prices of a timer call option in the Heston model. The constant interest rate is denoted by r, the strike of the call option is equal to $K = 100$. The initial stock price is $S_0 = 100$, and the parameters of the volatility dynamics (5.5) are given by $V_0 = 0.0625$, $\kappa = 2$, $\theta = 0.0324$ and $\sigma_v = 0.1$ The variance budget is determined by $\mathbb{V} = 0.0265$. In the last column, standard deviations are in parenthesis.

Panel A $r = 0\%$

Interest Rate r	Nb of Simul. n	Discret. Step $1/p$	Correl. Coeff. ρ	Monte Carlo Value (5.20)
0%	500000	1/500	$\rho = -0.8$	6.5073 (0.015)
0%	500000	1/500	$\rho = 0$	6.4871 (4.7e-014)
0%	500000	1/500	$\rho = 0.8$	6.4415 (0.015)
0%	500, 000	1/5000	$\rho = -0.8$	6.4786 (0.015)
0%	500, 000	1/5000	$\rho = 0$	6.4871 (4.7e-014)
0%	500, 000	1/5000	$\rho = 0.8$	6.4832 (0.015)
0%	10, 000, 000	1/500	$\rho = -0.8$	6.5224 (0.0033)
0%	10, 000, 000	1/500	$\rho = 0$	6.4871 (2.4e-013)
0%	10, 000, 000	1/500	$\rho = 0.8$	6.4651 (0.0033)
0%	10, 000, 000	1/5000	$\rho = -0.8$	6.4838 (0.0033)
0%	10, 000, 000	1/5000	$\rho = 0$	6.4871 (2.4e-013)
0%	10, 000, 000	1/5000	$\rho = 0.8$	6.4876 (0.0033)

Panel B $r = 4\%$

Interest Rate r	Nb of Simul. n	Discret. Step $1/p$	Correl. Coeff. ρ	Monte Carlo Value (5.20)
4%	500, 000	1/500	$\rho = -0.8$	7.6787 (0.016)
4%	500, 000	1/500	$\rho = 0$	7.5361 (0.00026)
4%	500, 000	1/500	$\rho = 0.8$	7.4002 (0.016)
4%	50, 0000	1/5000	$\rho = -0.8$	7.617 (0.016)
4%	50, 0000	1/5000	$\rho = 0$	7.5342 (0.00026)
4%	50, 0000	1/5000	$\rho = 0.8$	7.4364 (0.016)
4%	10, 000, 000	1/500	$\rho = -0.8$	7.6767 (0.0036)
4%	10, 000, 000	1/500	$\rho = 0$	7.5362 (5.9e-005)
4%	10, 000, 000	1/500	$\rho = 0.8$	7.4085 (0.0035)
4%	10, 000, 000	1/5000	$\rho = -0.8$	7.6344 (0.0036)
4%	10, 000, 000	1/5000	$\rho = 0$	7.5341 (5.9e-005)
4%	10, 000, 000	1/5000	$\rho = 0.8$	7.4324 (0.0035)

Table 5.2: Prices of a timer call option in the Heston model. The constant interest rate is denoted by r, the strike of the call option is equal to $K = 100$. The initial stock price is $S_0 = 100$, and the parameters of the volatility dynamics (5.5) are given by $V_0 = 0.0625$, $\kappa = 2$, $\theta = 0.0324$ and $\sigma_v = 0.1$ The variance budget is determined by $\mathbb{V} = 0.0265$. Prices are obtained using the approximation by Antonelli and Scarlatti (2009) given by (5.30) and (5.31).

Interest Rate r	Nb of Simul. n	Discret. Step $1/p$	Correl. Coeff. ρ	Monte Carlo Value (5.30)
0%	50, 000	1/500	$\rho = -0.8$	6.4624 (2.3e-005)
0%	50, 000	1/500	$\rho = 0.8$	6.5119 (2.3e-005)
0%	50, 000	1/5000	$\rho = -0.8$	6.4625 (2.3e-005)
0%	50, 000	1/5000	$\rho = 0.8$	6.5118 (2.3e-005)
4%	50, 000	1/500	$\rho = -0.8$	7.6227 (0.001)
4%	50, 000	1/500	$\rho = 0.8$	7.4492 (0.00066)
4%	50, 000	1/5000	$\rho = -0.8$	7.6211 (0.00099)
4%	50, 000	1/5000	$\rho = 0.8$	7.4482 (0.00066)

Table 5.3: Prices of a timer call option in the Hull and White model. The constant interest rate is denoted by r, the strike of the call option is equal to $K = 100$. The initial stock price is $S_0 = 100$, and the parameters of the volatility dynamics (5.6) are given by $\xi_v = 0.1$, $\mu_v = 0.2$. The variance budget is determined by $\mathbb{V} = 0.0265$. Prices are obtained using the approximation by Antonelli and Scarlatti (2009) given by (5.30) and (5.33).

Interest Rate r	Nb of Simul. n	Discret. Step $1/p$	Correl. Coeff. ρ	Monte Carlo Value (5.30)
0%	50000	1/3000	$\rho = -0.8$	6.459 (7.8e-008)
0%	50000	1/3000	$\rho = 0$	6.4871 (3.3e-015)
0%	50000	1/3000	$\rho = 0.8$	6.5153 (7.8e-008)
4%	50000	1/3000	$\rho = -0.8$	7.38 (3.7e-006)
4%	50000	1/3000	$\rho = 0$	7.3097 (3.2e-006)
4%	50000	1/3000	$\rho = 0.8$	7.2394 (2.6e-006)

Chapter 6

Simulation of barrier options under Heston stochastic volatility model

This chapter presents an almost exact Monte Carlo scheme for simulating the prices of continuous European barrier options under the general stochastic volatility models. We assume that there is no correlation between the stock price and the volatility process and that the risk-free interest rate is 0. The method is illustrated by pricing barrier options under the Heston model and the Hull-White model. The key idea we use is Theorem 1.3.1, which utilizes the stochastic time change idea.

6.1 Introduction

Pricing path-dependent options when the underlying's volatility is stochastic is a difficult problem, even in the well-known Heston stochastic volatility model. There are already a few recent results about pricing "mild" path-dependent options in the Heston model. For example closed-form expressions for prices of forward starting options have been given by Kruse and Nogel (2005). Griebsch and Wystup (2008) provide semi-analytical expressions for the prices of discretely-sampled barrier options under the Heston stochastic volatility model.

We first assume that the constant risk-free rate is equal to $r = 0\%$, and that the underlying and its volatility are two uncorrelated stochastic processes. These two assumptions are standard in the literature, such as in Faulhaber (2002), Lipton (2001) and Griebsch and Pliz (2010). Although these two assumptions are quite restrictive, Faulhaber (2002) discusses why these assumptions are critical in pricing exotic derivatives under stochastic volatility models, and illustrates the idea with the pricing of double-barrier options under the Heston model.

6.2 Pricing call option under Heston model

The dynamics for the Heston model are

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t \left(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \right) \\ dV_t &= \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW_1(t) \end{aligned} \tag{6.1}$$

where $W_1(t)$ and $W_2(t)$ are independent.

Now if we integrate from 0 to T on both sides of the SDE governing the variance process in (6.1), we have

$$V_T - V_0 = \kappa\theta T - \kappa \int_0^T V_s ds + \sigma_v \int_0^T \sqrt{V_s} dW_1(s) \tag{6.2}$$

From equation (6.2), if we denote $I(T) = \int_0^T V_s ds$, we can solve for $\int_0^T \sqrt{V_s} dW_1(s)$ as

$$\int_0^T \sqrt{V_s} dW_1(s) = \frac{1}{\sigma_v} (V_T - V_0 - \kappa\theta T + \kappa I(T)) \tag{6.3}$$

Also note that by Dubins-Schwartz theorem (see Karatzas and Shreve (1991)(Theorem 3. 4. 6, page 174), we have

$$\int_0^T \sqrt{V_s} dW_2(s) = B_{I(T)} \tag{6.4}$$

where B is a standard Brownian motion.

Proposition 6.2.1. *The price of the call option under Heston model is given by*

$$C_0 = E^Q \left[S_0 e^{A(V_T, I(T)) + \frac{1}{2}(1-\rho^2) I(T)} \mathcal{N}(d1) - K e^{-rT} \mathcal{N}(d2) \right] \tag{6.5}$$

where

$$d1 = d1(V_T, I(T)) = \frac{\ln\left(\frac{S_0}{K}\right) + rT + A(V_T, I(T)) + (1-\rho^2) I(T)}{\sqrt{(1-\rho^2) I(T)}}$$

$$d2 = d2(V_T, I(T)) = d1 - \sqrt{(1-\rho^2) I(T)} \tag{6.6}$$

Proof From the stock price dynamic in equation (6.2), we have

$$S_T = S_0 \exp\{rT - \frac{1}{2} I(T) + \rho \int_0^T \sqrt{V_s} dW_1(s) + \sqrt{1-\rho^2} \int_0^T \sqrt{V_s} dW_2(s)\}$$

$$= S_0 \exp\{rT - \frac{1}{2} I(T) + \frac{\rho}{\sigma_v} (V_T - V_0 - \kappa\theta T + \kappa I(T)) + \sqrt{1-\rho^2} B_{I(T)}\}$$

$$= S_0 \exp\{rT + \frac{\rho}{\sigma_v} V_T + \left(\frac{\kappa\rho}{\sigma_v} - \frac{1}{2}\right) I(T) - \frac{\rho}{\sigma_v} V_0 - \frac{\kappa\theta\rho}{\sigma_v} T + \sqrt{1-\rho^2} B_{I(T)}\}$$

If we define the function $A(V_T, I(T))$ as

$$A(V_T, I(T)) = \frac{\rho}{\sigma_v} V_T + \left(\frac{\kappa\rho}{\sigma_v} - \frac{1}{2}\right) I(T) - \frac{\rho}{\sigma_v} V_0 - \frac{\kappa\theta\rho}{\sigma_v} T \quad (6.7)$$

Then we have

$$S_T = S_0 \exp\{rT + A(V_T, I(T)) + \sqrt{1-\rho^2} B_{I(T)}\} \quad (6.8)$$

We can see from equation (6.8) that S_T is a function of V_T and $I(T)$. That motivates us to price the option by first conditioning on $(V_T, I(T))$. The price of the call option C_0 is given by

$$\begin{aligned} C_0 &= E^Q\left[e^{-rT}(S_T - K)^+\right] \\ &= E^Q\left[E^Q\left[e^{-rT}(S_T - K)^+ \mid V_T, I(T)\right]\right] \quad (6.9) \end{aligned}$$

Note that from the equation (6.8), we know that conditional on the values of $(V_T, I(T))$, S_T is just log-normally distributed,

$S_T \sim Lognormal\left(\ln(S_0) + rT + A(V_T, I(T)), (1-\rho^2)I(T)\right)$

So we can use the Black-Scholes formula to calculate the inner expectation in (6.9) and we have proved the proposition 6.2.1. $\square$

So our new algorithm is

Algorithm

For each of the M Monte Carlo paths,

Step 1: Simulate $(V_T, I(T))$ jointly.

Step 2: Conditional on the value of $(V_T, I(T))$, we record the value of the call from equation (6.5).

Now the key problem is to simulate $(V_T, I(T))$ jointly. This is detailed in section 6.2.1.

6.2.1 Joint MGF of $(V_T, I(T))$

The exact simulation of $(V_T, I(T))$ is made possible because we can derive the closed-form joint moment generating functions of $(V_T, I(T))$. Define

$$M(u, v) = E^Q\left[\exp\left(uV_T + vI(T)\right)\right] \quad (6.10)$$

This joint mgf can be obtained based on a recent result by Hurd and Kuznetsov(2006). Based on Theorem 3.1 in Hurd and Kuznetsov(2006), if we take $d_2 = 0$ and $w_2 = 0$, then we have

$$M(u,v) \quad = \quad G^{CIR}(T, V_0; -v, 0, -u, 0) \tag{6.11}$$

We introduce some notations similar to Hurd and Kuznetsov(2006) before we proceed to calculate $G^{CIR}(T, V_0; -v, 0, -u, 0)$. Denote $a = \kappa\theta$, $\alpha = \frac{2\kappa\theta}{\sigma_v^2} - 1$, $\beta = \frac{2\kappa}{\sigma_v^2}$ $v_1 = \frac{1}{2}\left(-\beta + \sqrt{\beta^2 - \frac{8v}{\sigma_v^2}}\right)$ $\gamma_T = (\beta + 2v_1)\left(1 - e^{-(\frac{\beta}{2}+v_1)\sigma_v^2 T}\right)^{-1}$. Then we have

$$\begin{aligned} M(u,v) &= G^{CIR}(T, V_0; -v, 0, -u, 0) \\ &= e^{-av_1T}\left(\frac{\gamma_T}{\gamma_T - u - v_1}\right)^{\alpha+1} \\ &\exp\left(-V_0\left(v_1 - \frac{\gamma_T(u+v_1)}{\gamma_T - u - v_1}e^{-\left(\frac{\beta}{2}+v_1\right)\sigma_v^2 T}\right)\right) \end{aligned} \tag{6.12}$$

From equation (6.12), we have

$$M(u,0) = (\frac{\gamma_T}{\gamma_T - u})^{\alpha+1}\exp\{V_0\frac{\gamma_T u}{\gamma_T + u}e^{-\frac{\beta\sigma_v^2 T}{2}}\} \tag{6.13}$$

$$M(0,v) = e^{-av_1T}\left(\frac{\gamma_T}{\gamma_T - v_1}\right)^{\alpha+1}\exp\left(-V_0\left(v_1 - \frac{\gamma_T v_1}{\gamma_T - v_1}e^{-\left(\frac{\beta}{2}+v_1\right)\sigma_v^2 T}\right)\right)$$

From this joint moment generating function, we can also get the joint Laplace transform by changing u and v to be $-u$ and $-v$. Thus we can invert this joint Laplace transform to get the joint density function of $(V_T, I(T))$ and then construct a three dimensional acceptance-rejection algorithm to simulate $(V_T, I(T))$ exactly.

6.3 Pricing of call option under Heston model by Fourier transform

We can make use of the joint moment generating function in equation (6.12) directly in pricing call options under the Heston model by using the idea in Borovkov and Novikov (2002). Then we can give a **closed-from** formula for the price of the call option under Heston model. Note that the formula we obtain here is new and it is simpler than the original Heston (1993) formula, since here our formula involves just one integral term while the original Heston formula requires evaluation of **two** integral terms. We have the following proposition.

Proposition 6.3.1. *The price of the call option under Heston model is*

$$C_0 = \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} \Phi(1 + a - is) g(s) ds \tag{6.14}$$

where

$$g(s) = \frac{e^{isb-ab}}{(a - is)(1 + a - is)} \tag{6.15}$$

and $b = \ln(K)$,

$$\Phi(u) = S_0^u e^{\left(rT - \frac{\rho}{\sigma_v} V_0 - \frac{\kappa\theta\rho}{\sigma_v} T\right)u} M\left(\frac{\rho}{\sigma_v} u, \frac{\kappa\rho}{\sigma_v} u - \frac{1}{2} u + \frac{1}{2}(1 - \rho^2) u^2\right) \tag{6.16}$$

Proof From the Theorem 1 in Borovkov and Novikov (2002), we only need to calculate $\Phi(u) = E\left[e^{u \ln(S_T)}\right]$. From the equation (6.8), we have

$$\begin{aligned}
\Phi(u) &= E\left[e^{u\ln(S_T)}\right] \\
&= E\left[S_0^u e^{urT + uA(V_T, I(T)) + u\sqrt{1-\rho^2} B_{I(T)}}\right] \\
&= E\left[E\left[S_0^u e^{urT + uA(V_T, I(T)) + u\sqrt{1-\rho^2} B_{I(T)}} \mid (V_T, I(T))\right]\right] \\
&= S_0^u e^{\left(rT - \frac{\rho}{\sigma_v} V_0 - \frac{\kappa\theta\rho}{\sigma_v} T\right)u} E\left[\exp\left(\left(\frac{\rho}{\sigma_v} u\right) V_T + \left(\frac{\kappa\rho}{\sigma_v} u - \frac{1}{2} u + \frac{1}{2}(1 - \rho^2) u^2\right) I(T)\right)\right] \\
&= S_0^u e^{\left(rT - \frac{\rho}{\sigma_v} V_0 - \frac{\kappa\theta\rho}{\sigma_v} T\right)u} M\left(\frac{\rho}{\sigma_v} u, \frac{\kappa\rho}{\sigma_v} u - \frac{1}{2} u + \frac{1}{2}(1 - \rho^2) u^2\right) \qquad (6.17)
\end{aligned}$$

Then proof is done. □

Remark Proposition 6.3.1 gives us a single integral expression of the call option price under the Heston model. Also note that we can similarly price all the **power options** under Heston model using the formula given in Proposition 6.3.1. This is not a trivial problem if we base analysis on the original Heston (1993) formula, but it is easy to be derived from our formula. The result is given in the next proposition.

Proposition 6.3.2. *For the power call option with payoff* $(S_T^p - K)^+$*, its price is given as*

$$C_0 = \frac{e^{-rT}}{2\pi}\int_{-\infty}^{\infty}\Phi_p(1+a-is)g(s)ds \tag{6.18}$$

where $g(s)$ *is given in* (6.15). *and*

$$\Phi_p(u) = S_0^{pu}e^{\left(rT-\frac{\rho}{\sigma_v}V_0-\frac{\kappa\theta\rho}{\sigma_v}T\right)pu}M\left(\frac{\rho}{\sigma_v}pu, \frac{\kappa\rho}{\sigma_v}pu - \frac{1}{2}pu + \frac{1}{2}(1-\rho^2)p^2u^2\right) \tag{6.19}$$

Proof It can be easily verified and proof is thus omitted. □

6.4 Pricing of Continuous Barrier Options under Stochastic Volatility

We consider barrier options written on an underlying stock S in an arbitrage-free market.

Model for the stock price

Let S_t denote the stock price at time t and let $\lambda(V_t)$ be a functional of its volatility. Assuming that the risk-free interest rate is equal to 0 and that the stock price is not correlated to the volatility process, we consider the following general stochastic volatility model under a given risk-neutral probability Q

$$\begin{cases} dS_t &= \lambda(V_t)\,S_t\,dW_t^1 \\ dV_t &= \mu(t,V_t)\,dt\;+\;v(t,V_t)\,dW_t^2 \end{cases} \tag{6.20}$$

where W^1 and W^2 are two independent standard Brownian motions under Q, and λ, μ and v are deterministic functions of the time t and the variance V_t. Under these conditions, the underlying stock price can also be written as

$$S_t = S_0\exp\left\{-\frac{1}{2}\int_0^t\lambda^2(V_s)ds + \int_0^t\lambda(V_s)dW_s^1\right\}$$

By the Dubins-Schwartz theorem (see Karatzas and Shreve (1991)(Theorem 3. 4. 6, page 174), the martingale $\int_0^t \lambda(V_s)dW_s^1$ can be written as a time-changed Brownian motion

$$\int_0^t \lambda(V_s)dW_s^1 = B_{I_t}, \text{ where } I_t = I(0,t) = \int_0^t \lambda^2(V_s)ds, \tag{6.21}$$

and where B is a standard Brownian motion. Thus we obtain the following expression for S_t

$$S_t = S_0 \exp\left\{-\frac{1}{2}I_t + B_{I_t}\right\}. \tag{6.22}$$

First Hitting Time Options

Our method can be applied to pricing a wide class of exotic options linked to the first time a certain condition is met. It could be the first passage time to a given level for the continuously monitored process or for a discretely monitored process. It could also be a double-barrier condition (see Davydov and Linetsky (2001), Geman and Yor (1993), Kolkiewicz (2002)). Recall that a barrier option has a payoff linked to the first hitting time to a given threshold L. Let τ be the first passage time of the stochastic process S_t given in (6.22) to the threshold L.

A standard European barrier option has a payoff paid at a fixed maturity time T in the future depending on the fact whether τ happens or not before time T. We modify the standard barrier option design by assuming that the option is exercised when the barrier condition is met rather than at maturity. Similar as Bernard and Boyle (2010), we call them E^2 (early exercise) barrier options. Note that the trajectory of the stock price is continuous in our setting. For example in the case of a perpetual up and in E^2-barrier call option continuously monitored with up barrier L and strike $K < L$, its payoff would be equal to $L - K$ paid at the first hitting time τ. In the case of a barrier option discretely monitored, S_τ can be different from L.

We divide our analysis in two parts. We first study the easier case of an E^2-option whose payoff is paid at the random time τ and depends on S_τ. Thus the payoff of an E^2-option can be expressed as $f(\tau, S_\tau)$. It includes the case when the payoff is paid only when τ is prior to T (the payoff function includes $\mathbb{1}_{\tau<T}$). Second, we investigate standard barrier options whose payoffs are paid at a fixed time T, and depend on τ and S_T (or more generally can also depend on the maximum of S over $[0,T]$). The payoff of this option is $g(\tau, S_T)$ or $g(\tau, \max_{t\in[0,T]}(S_t))$. This payoff is a generic payoff that include the cases of all kind of barrier options, lookback options and combination of barrier and lookback features. For example, the Up-and-In barrier call option has the following payoff $(S_T-K)^+\mathbb{1}_{\{\tau<T\}}$, where τ is the first passage time of S_t to an up level $H>S_0$.

6.4.1 E^2-Options with payoff $f(\tau, S_\tau)$

Recall that I_t defined by (6.21) denotes the stochastic clock. Conditioned on I_t, the expression for S_t in (6.22) is similar to the one of a standard Geometric Brownian motion (GBM hereafter). Since the variance process V_t and the stock price S_t are independent, I_t and B_{I_t} are also independent.

Algorithm 1 For each Monte Carlo path,
Step 1: Simulation of $\hat{I}$ under the stochastic clock. *Under the stochastic clock, simulate the stopping time $\hat{I}$ such that the condition of the activation/deactivation of the option is satisfied. Note that this time $\hat{I}$ is the time measured under the* ***stochastic clock****.* When the stock price is a GBM, the distribution of the stopping time τ for standard barrier, double barrier options is known (it could be through its Laplace transform, Fourier transform). Since the stock price is a GBM under the stochastic clock, it is straightforward to simulate τ under the stochastic clock. Let us denote it by $\hat{I}$. For example, in the barrier option case, $\hat{I}=\inf\{u>0, S_0\exp\{-\frac{1}{2}u+B_u\}\geqslant L\}$. We know the exact distribution of this first passage time $\hat{I}$ because it can be reduced to the problem of the first passage time for a drifted Brownian motion (its distribution is given by formula 2. 02, page 295, Borodin and Salminen(2002), or by formula (29) in Appendix 2 of Bernard and Boyle (2010)). It is distributed as Inverse Gaussian distribution.

Step 2: Simulation of τ. *Given the simulated value $\hat{I}$ in Step 1, we obtain the time under the **original clock** such that the condition of the activation/deactivation of the option is met.* Let us denote the time measured under the original clock as τ, then we have the following property

$$\tau = \inf \left\{ t > 0, \int_0^t \lambda^2(V_s)ds \geqslant \hat{I} \right\}. \tag{6.23}$$

Another way to interpret this time is that τ is the first passage time of the integrated variance process to a fixed level $\hat{I}$. The law of (τ, V_τ) is known in very general volatility models. It is given in the Theorem 1.3.1.

In the stochastic volatility models for which it is possible to simulate the process X_t solution to the SDE given by (1.3), and (1.2) we then obtain τ and V_τ. This will be discussed in two well-known stochastic volatility models: the Heston model and the Hull and White model. Theorem 1.3.1 is especially useful in the pricing of timer options. Interested readers can also refer to Carr and Lee (2010), Li (2009b) and Bernard and Cui (2010).

Note that by definition we have $S_\tau = L$, so from the above procedure, we can simulate jointly τ and S_τ and obtain the price of any derivatives with payoffs $f(\tau, S_\tau)$ using the expression of the price as the expectation under the risk-neutral probability (since the risk-free rate is $r = 0$).

$$C_0 = E^Q \left[f(\tau, S_\tau) \right] \tag{6.24}$$

Note that this Monte Carlo algorithm is *almost exact* and the only part that needs discretization is the Step 2 when we simulate τ. Numerical examples will be given in section 6.5.

Theorem 1.3.1 for specific stochastic volatility models

Heston model

In the special case of Heston Stochastic volatility model,

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t \tag{6.25}$$

we have

$$(\tau, V_\tau) \sim^{law} \left(\int_0^\eta \frac{ds}{\sigma_v X_s}, \ \sigma_v X_\eta \right), \tag{6.26}$$

where X_t is a standard Bessel process governed by the following SDE

$$dX_t = \left(\frac{\kappa\theta}{\sigma_v^2 X_t} - \frac{\kappa}{\sigma_v}\right) dt + dB_t,\ X_0 = \frac{V_0}{\sigma_v}, \tag{6.27}$$

where B denotes a standard Brownian motion. A proof of (6.26) is presented in Appendix 6.6.

Hull and White model

In the special case of Hull and White volatility model, we have

$$dV_t = \mu_v V_t dt + \sigma_v V_t dW_t \tag{6.28}$$

we have

$$(\tau, V_\tau) \sim^{law} \left(\frac{4}{\xi_v^2}\int_0^\eta \frac{ds}{Y_s^2},\ \frac{\xi_v^2}{4} Y_\eta^2\right), \tag{6.29}$$

where Y_t is governed by the following SDE

$$dY_t = \left(\frac{2\mu_v}{\xi_v^2} - \frac{1}{2}\right)\frac{1}{Y_t} dt + dB_t,\ Y_0 = \frac{2}{\xi_v}\sqrt{V_0}, \tag{6.30}$$

where B denotes a standard Brownian motion. We can immediately see that Y_t is a standard Bessel process with index $v = \frac{2\mu_v}{\xi_v^2} - 1$. A proof of (6.29) is presented in Appendix 6.7.

6.4.2 European Options with payoff $g(\tau, S_T)$

From the ideas of the previous section, we already know how to simulate jointly (τ, S_τ, V_τ). Conditional on $\mathcal{F}_\tau$, the stock price S_T can be simulated. Note that τ is a stopping time which is $\mathcal{F}_\tau$ measurable. Using the strong Markov property, the simulation of S_T conditioned by the current knowledge at time τ does not depend on any past information. From (6.22), the stock price at maturity is

$$S_T = S_\tau \exp\left\{-\frac{1}{2} I(\tau, T) + B_{I(\tau,T)}\right\} \tag{6.31}$$

where

$$I(\tau, T) = \int_\tau^T V_s ds. \tag{6.32}$$

The first step is then to simulate $I(\tau, T)$ conditional on τ, S_τ and V_τ. It is then immediate to simulate S_T because S_T is log normally distributed conditional on the value of $I(\tau, T)$.

6.4.3 Simulation of $I(\tau, T)$

There are two possible methods. The first one is developed by Broadie and Kaya (2006). It needs two steps

1. Simulate V_T conditional on V_τ using the fact that $V_T \mid V_\tau$ is non-central Chi-square distributed.

2. Simulate $I(\tau, T)$ conditional on V_τ and V_T. In Broadie and Kaya (2006), the characteristic function of $I(\tau, T) \mid V_\tau, V_T$ is given. An inversion of the Fourier transform is then implemented.

This can be computationally challenging and time consuming because the characteristic function involves modified Bessel functions. Two more recent developments improve Broadie and Kaya (2006). Glasserman and Kim (2010) proposed a Gamma expansion of the conditional distribution and give an equivalence in law. Smith (2007) proposes an approximation which allows for storing inverted values in a cache to speed up the simulation method.

In our case, we do not need to make use of this two-step technique to simulate $I(\tau, T)$ because we do not need V_T. The problem of simulating $\int_\tau^T V_s ds \mid V_\tau$ is actually easier than simulating $\int_\tau^T V_s ds \mid V_\tau, V_T$. In several well-known stochastic volatility models, the moment generating function (or the Laplace transform) of $I(\tau, T)$ is known[1]. Using Abate and Whitt (1995) to invert the Laplace transform, we obtain the distribution of $I(\tau, T)$ and can easily implement an acceptance rejection method. For example in the case of the Heston model, the moment generating function of $I(t, T)$ is linked to the well-known risk-free bond formula in the CIR (Cox Ingersoll

[1]The moment generating function of $I(\tau, T)$ can be interpreted as the price of a risk free bond where the instantaneous risk-free interest rate would be V_s. See for example Ball and Roma(1994).

Ross) model. In fact in the Heston model, the Hull and White model and the Stein-Stein model, the moment generating function of $I(t,T)$ is known (see section 6.8). This method is therefore straightforward, exact and efficient.

To summarize, in order to obtain the payoff $g(\tau, S_T)$, and calculate the price of a standard barrier option,

$$C_0 = E^Q\left[g(\tau, S_T)\right]. \tag{6.33}$$

We may simulate (τ, S_T) jointly according to the following steps.

Algorithm 2

Implement Steps **1**, and **2** presented in Algorithm 1 in section 6.4.1.

Step 3 Simulate $I(\tau,T)$ conditional on V_τ following the second method presented above.

Step 4 Conditional on $I(\tau,T)$, simulate S_T based on (6.31).

For efficient numerical implementation. we use acceptance-rejection sampling method based on the density of $I(\tau,T)$, which can be obtained based on Ju and Zhong(2006). From equation (17) and (21) in Ju and Zhong(2006), we have the following density function for $y = I(\tau,T)$

$$f(y) = \frac{2}{\pi}\int_0^\infty \cos(\eta y)\Re\{F(V_\tau, \eta, T-\tau)\}d\eta \tag{6.34}$$

Here

$$F(V_\tau, \eta, T-\tau) = A(T-\tau)e^{-i\eta B(T-\tau)V_\tau} \tag{6.35}$$

where

$$\begin{aligned} A(T-\tau) &= \left(\frac{2v\exp\{\frac{(\kappa+v)(T-\tau)}{2}\}}{(\kappa+v)(\exp\{v(T-\tau)\}-1)+2v}\right)^{\frac{2\kappa\theta}{\sigma_v^2}} \\ B(T-\tau) &= \frac{2(\exp\{v(T-\tau)\}-1)}{(\kappa+v)(\exp\{v(T-\tau)\}-1)+2v} \\ v &= \sqrt{\kappa^2 + 2i\eta\sigma_v^2} \end{aligned} \tag{6.36}$$

Note that when $\frac{2\kappa\theta}{\sigma_v^2}$ is not an integer, $A(T-\tau)$ is multi-valued and we should choose its principal branch.

Note that the European Barrier option under Heston model belongs to this class of payoffs since its payoff is a function of τ and S_T. To simulate its price, we have

Algorithm 3 (for Up and In case)

Implement Steps **1**, and **2** presented in Algorithm 1 in section 6.4.1.

Step 3 If $\tau > T$, it is deactivated, then we record payoff 0 for this Monte Carlo path. If $\tau \leqslant T$, it is activated and we can use the original Heston formula to calculate the call option price and record this value for this Monte Carlo path.

Step 4 Then we average out the payoffs along each Monte Carlo path and then can get the price of the Up and In barrier call under the Heston model.

Remark The Up and Out case and also the pricing of barrier options under Hull-White model can be dealt with similarly. Next section provides some numerical examples.

6.5 Numerical Examples

In this section we give some numerical examples.

Table 6.1: Prices of a continuous Up and In barrier call option in the Heston model. The interest rate is $r = 0$, and the correlation is $\rho = 0$. The strike of the call option is equal to $K = 100$. The initial stock price is $S_0 = 100$, the barrier level is $L = 120$ and the maturity time is $T = 1$. The parameters of the volatility dynamics (6.25) are given by $\kappa = 2$, $\theta = 0.0324$, and $\sigma_v = 0.1$.

Nb of Discretization	Monte Carlo runs	Up In Call	Vanilla Call	Time(seconds)
100	1000	4.4178	7.7235	9.946068
100	5000	4.2216	7.7235	49.720493
100	10000	4.2054	7.7235	75.342871
100	50000	4.1674	7.7235	331.351235

Table 6.2: Prices of a continuous Up and Out barrier call option in the Heston model. The interest rate is $r = 0$, and the correlation is $\rho = 0$. The strike of the call option is equal to $K = 100$. The initial stock price is $S_0 = 100$, the barrier level is $L = 120$, and the maturity time is $T = 1$. The parameters of the volatility dynamics (6.25) are given by $\kappa = 2$, $\theta = 0.0324$, and $\sigma_v = 0.1$.

Nb of Discretization.	Monte Carlo runs.	Up Out Call.	Vanilla Call	Time(seconds)
100	1000	3.3056	7.7235	9.946068
100	5000	3.5018	7.7235	49.720493
100	10000	3.5180	7.7235	75.342871
100	50000	3.5560	7.7235	331.351235

We can see from the outputs of the table that the price of Up and In Barrier call and the price of the Up and Out call sum up to the price of the Vanilla call under the Heston model. This agrees with the barrier in out parity, which confirms our result.

6.6 Proof of Formula (6.26)

For Heston Stochastic Volatility model, recall equation (6.20), we have correspondingly, $\mu V_t = \kappa(\theta - V_t)$ and $v(V_t) = \sigma_v V_t$, then make use of Theorem 1.3.1, we can arrive at the result (6.26). To see another straightforward proof, refer to Section 5.5.

6.7 Proof of Formula (6.29)

For Hull-White Stochastic Volatility model, recall equation (6.20), we have correspondingly, $\mu(V_t) = \mu_v V_t$ and $v(V_t) = \sigma_v V_t$, then make use of Theorem 1.3.1, we can arrive at the result (6.29). Note that here $Y_t = \frac{2}{\xi_v}\sqrt{X_t} = \frac{2}{\xi_v}\sqrt{V_{\tau_t}}$. To see another straightforward proof, refer to Section 5.6.

6.8 Moment generating function of $I(t,T)$ in Heston model

In this appendix we recall the formula for the moment generating function of $I(t,T)$. From the idea in Ball and Roma(1994), we can link the moment generating function of $I(\tau,T)$ to the risk-free bond formula. The MGF $m(u) = E\left[e^{uI(\tau,T)}\right]$ is

$$m(u) = E\left[e^{uI(\tau,T)}\right] = A(T-\tau)e^{uB(T-\tau)V_\tau} \tag{6.37}$$

where

$$\begin{aligned} A(T-\tau) &= \left(\frac{2v\exp\{\frac{(\kappa+v)(T-\tau)}{2}\}}{(\kappa+v)(\exp\{v(T-\tau)\}-1)+2v}\right)^{\frac{2\kappa\theta}{\sigma_v^2}} \\ B(T-\tau) &= \frac{2(\exp\{v(T-\tau)\}-1)}{(\kappa+v)(\exp\{v(T-\tau)\}-1)+2v} \\ v &= \sqrt{\kappa^2 - 2u\sigma_v^2} \end{aligned} \tag{6.38}$$

We can easily see that this moment generating function is only defined in the interval $u \leqslant \frac{\kappa^2}{2\sigma_v^2}$.

6.9 Conclusion of Chapter 6

In this chapter we discuss the pricing of continuous barrier options under the Heston stochastic volatility model. We make use of the "stochastic time-change" idea and reduce the pricing problem into a two-step procedure. We first simulate the first hitting time I of a standard drifted Brownian motion under the "new clock" and then simulate the corresponding time of this time measured under the "old clock" τ. Then we know that τ is the first hitting time we want for the original time-changed process to hit the fixed level η. This idea is also employed in Hurd and Kuznetsov(2006), where they deal with first hitting time of the Brownian motion time-changed by an independent Levy subordinator.

Conditional on the simulated first hitting time τ and the variance process value at this time V_τ, we can carry on Monte Carlo simulations to price continuous barrier options. This method can also be applied to pricing other options related to the first passage time of the underlying process and this is left as further research.

Chapter 7

Conclusion and further research direction

In this thesis I have illustrated the application of time-change method in solving different option pricing problems in quantitative finance. There are two types of time changes involved: the deterministic time change and the stochastic time change.

The discounted asset price process under Hull-White stochastic interest rate model can be expressed as the exponential of time-changed Brownian motion with deterministic time change. This allows us to use this expression for the pricing of exotic options under stochastic interest rate. This idea is well illustrated in Chapter 3. Since indicators involving the length of time intervals are not well-preserved under state to state transform from the "old time" to the "new time", we can not price **exactly** exotic derivatives involving occupation time and excursion time of the underlying asset process. However, we have inequality conditions and find lower or upper bounds for the desired price. This idea is illustrated in Chapter 4. Further research may be in the direction of pricing barrier option under stochastic interest rates with constant barrier, which is very useful in modeling corporate default over long time horizons.

The stochastic time change is a popular way to construct Stochastic volatility models. The popular Variance Gamma model is just the Brownian motion subordinated by an independent Gamma process. The timer option we consider is one of the "Quadratic-variation based derivative products". It is also named "Mileage Option". Some other products are variance swaps and volatility swaps. We make use of the stochastic time change idea and it provides us with a unique viewpoint of the problem. Further research can be in the direction of connecting the pricing of "timer option" to the pricing and hedging of other exotic volatility products. Another direction for future research is to **exactly** simulate the price of timer option bias free. Pricing continuous barrier option under Heston stochastic volatility model can be quite difficult and there is little literature on this. We attack the problem from first passage time perspective. Stochastic time change can also be helpful when we study the pricing of exotic path-dependent options under Stochastic Volatility models. Chapter 6 illustrates its use in pricing barrier options under stochastic volatility models. Note that we make the rather restrictive assumption that the interest rate $r = 0$ and the correlation $\rho = 0$. This is standard assumption in the literature dealing with barrier option pricing under stochastic volatility models. Further research direction might be to relax these two assumptions.

Above all, the key idea in applying time change method is that the asset price is distributed as Geometric Brownian motion conditional on the time change. Thus we can first conduct the time change, solve the problem under the *new* time, and then carry the solution back to the *old* time and solve the problem in full.

The time change method can be useful in many other areas in quantitative finance. I believe that its power has not been fully exploited and this is left for future research.

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